Multilevel Preconditioning for Isogeometric Analysis

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Setup

- Elliptic PDE on physical domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$
- Isogeometric analysis $\mathbf{F} : \hat{\Omega} \to \Omega$ for parametric domain $\hat{\Omega} \subset [0, 1]^d$ with regular mapping $\mathbf{F}$

Goal: High order approximate solution with fast iterative method

Ingredients

- isogeometric discretization by tensor product B-splines of degree p
- iterative solution with multilevel preconditioner and nested iteration

$\rightsquigarrow$ solution in optimal linear complexity

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Problem Setup

Elliptic PDE of order $2r$ on physical domain Ω

$$
\begin{array}{llll}
r = 1: & -\Delta u = f & \text{in } \Omega, & u|_{\partial\Omega} = 0 \\
r = 2: & \Delta^2 u = f & \text{in } \Omega, & u|_{\partial\Omega} = \mathbf{n} \cdot \nabla u|_{\partial\Omega} = 0
\end{array}
$$

Weak operator form: for given $f \in H^{-r}(\Omega)$, find $u \in H_0^r(\Omega)$ such that

$$
\boxed{Au = f \quad \text{in} \quad H^{-r}(\Omega)}
$$

Elliptic operator A defined by $\langle Av, w \rangle := a(v, w)$ symmetric, continuous

and coercive on $H_0^r(\Omega)$: $\quad \|Av\|_{H^{-r}(\Omega)} \sim \|v\|_{H^r(\Omega)}$

Mapping $\mathbf{F} : \hat{\Omega} \to \Omega$ from parametric domain $\hat{\Omega} \subset [0, 1]^d$ to physical domain Ω

$\mathbf{F}$ regular: $\quad \|D\mathbf{F}(\mathbf{x})\| \sim 1$ for all $\mathbf{x} \in \hat{\Omega}$

Numerical Solution on Physical Space

Discretization on uniform grid: $V_h \subset H_0^r(\Omega)$ $\quad$ $\dim V_h < \infty$ $\quad \rightsquigarrow \quad$ $\boxed{A_h\, u_h = f_h \qquad (*)}$

$0 < h < 1$ grid size

Goal: Realize discretization error accuracy ε
with minimal amount of work $\mathcal{O}(N(\varepsilon))$ in amount of unknowns $N(\varepsilon)$

Obstructions for fast numerical solution:

- Large sparse linear system of equations $(*)$ $\rightsquigarrow$ iterative solver
- Convergence speed of iterative solver depends on $\mathrm{cond}_2(A_h)$
- Standard discretizations with finite differences or finite elements $\rightsquigarrow$ $\mathrm{cond}_2(A_h) \sim h^{-2r}$
  $0 < h < 1$ grid size
- High desired accuracy, resolution of singularities in data and/or geometry $\rightsquigarrow$ small h
  $\rightsquigarrow$ larger problem $\rightsquigarrow$ worse condition number

Ingredients for reaching goal:

(i) Multilevel preconditioner C_h
multigrid methods, BPX preconditioner, wavelet discretizations $\rightsquigarrow$ $\mathrm{cond}_2(C_h A_h) \sim 1$

(ii) Nested iteration

A-priori Estimates for Finite Elements

Quality measure: Approximation in norm $\|u - u_h\|_{L_2(\Omega)} \leq \varepsilon$

A–priori error estimates: $\Omega \subset \mathbb{R}^d$ dim $V_h = N \sim h^{-d}$ uniform grid

$$\|u - u_h\|_{L_2(\Omega)} \lesssim h^s \|u\|_{H^s(\Omega)} \qquad u_h \in V_h \qquad 0 \leq s \leq p+1$$

$$\Longleftrightarrow \qquad \|u - u_N\|_{L_2(\Omega)} \lesssim N^{-s/d} \|u\|_{H^s(\Omega)}$$

N degrees of freedom $\longleftrightarrow$ accuracy $\mathcal{O}(N^{-(p+1)/d})$

Approximation rate determined by

(i) (piecewiese polynomials of degree $p \rightsquigarrow$) approximation order $p+1$ of V_h

(ii) space dimension d

(iii) amount of smoothness of u in L_2

Target:

Realize discretization error accuracy $\varepsilon \sim h^{p+1} \sim 2^{-(p+1)J}$ for grid with spacing $h \sim 2^{-J}$

Problem complexity: For $h \sim 2^{-J}$ a total of $N \sim 2^{Jd}$ unknowns

Optimal complexity for iterative solver: Minimal amount of work is $\mathcal{O}(N)$

Isogeometric Elements of Degree p

Mesh on $[0,1]$: $\Xi := \{\xi_1, \ldots, \xi_{n+p+1}\}$ a p-open knot vector such that

$$0 = \xi_1 = \ldots = \xi_{p+1} < \xi_{p+2} \leq \ldots \leq \xi_n < \xi_{n+1} = \ldots = \xi_{n+p+1} = 1$$

(allowing internal repeated knots having multiplicity $m_i \leq p - r + 1$)

B-splines of degree p on Ξ defined recursively by

$$p = 0: \quad N_{i,0}(\zeta) = \begin{cases} 1 & \text{if } \xi_i \leq \zeta < \xi_{i+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$p \geq 1: \quad N_{i,p}(\zeta) = \frac{\zeta - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\zeta) + \frac{\xi_{i+p+1} - \zeta}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\zeta)$$

$\rightsquigarrow$ n B-splines spanning spline space of piecewise polynomials of degree p
with $p - m_i$ continuous derivatives at inner nodes

$$S_h(\hat{\Omega}) := \text{span}\left\{ B_i(\mathbf{x}) := \prod_{\ell=1}^{d} N_{i_\ell,p}(x_\ell), \; i = 1, \ldots, N := nd \right\}$$

Mapping $\mathbf{F} = (F_1, \ldots, F_d)^T \quad F_i \in S_{\tilde{h}}(\hat{\Omega})$ for some $\tilde{h} \gg h$

$\rightsquigarrow V_h := \{ v_h \in H_0^r(\Omega) : \; v_h \circ \mathbf{F} \in S_h(\hat{\Omega}) \}$

$$\|v\|^2_{L_2(\Omega)} := \int_{\hat{\Omega}} |v(\mathbf{F}(\mathbf{x}))|^2 \|D\mathbf{F}(\mathbf{x})\| d\mathbf{x}$$

Theorem

(S) Uniform stability of basis with respect to $L_2(\Omega)$

$$\| \sum_{i=1}^{N} c_i \, B_i \circ \mathbf{F}^{-1} \|^2_{L_2(\Omega)} \sim \sum_{i=1}^{N} |c_i|^2 =: \|\mathbf{c}\|^2_{\ell_2} \qquad \text{for any } \mathbf{c} \in \ell_2 \quad \text{constants} = c(p,d) \neq c(h)$$

(J) Direct or Jackson estimates

$$\inf_{v_h \in V_h} \|v - v_h\|_{L_2(\Omega)} \lesssim h^s \, |v|_{H^s(\Omega)} \qquad \text{for any } v \in H^s(\Omega) \quad 0 \leq s \leq p+1$$

(B) Inverse or Bernstein estimates

$$\|v_h\|_{H^s(\Omega)} \lesssim h^{-s} \|v_h\|_{L_2(\Omega)} \qquad \text{for any } v_h \in V_h \text{ and } 0 \leq s \leq p$$

Multilevel Preconditioner

Asymptotically optimal preconditioner: C_h such that

$$\mathrm{cond}_2(C_h A_h) \sim 1$$

and setup and application of C_h in optimal linear complexity $\mathcal{O}(N)$

Schwarz iterative schemes based on subspace corrections

$\rightsquigarrow$ Multilevel schemes yielding optimal preconditioners:

- Multiplicative schemes $\rightsquigarrow$ multigrid methods — Brandt, Braess, Bramble, Hackbusch, Zulehner . . .
  IgA: Gahalaut, Kraus, Tomar . . .
- Additive schemes $\rightsquigarrow$ BPX preconditioner; wavelet discretization
  Bramble, Pasciak, Xu, Yserentant, Oswald, Dahmen, Kunoth . . .

Relevant idea from Approximation Theory: Multilevel characterization of function spaces and norm equivalences

Not optimal are preconditioners based on domain decomposition, overlapping Schwarz, hierarchical basis preconditioners. . . Beirao da Veiga, Cho, Pavarino. Scacci, Kleiss, Pechstein, Jüttler, Langer . . .

$V_h \longleftrightarrow V_j \qquad h \sim 2^{-j} \qquad j$ resolution level

Multiresolution $V_{j_0} \subset V_{j_0+1} \subset \ldots \subset V_j \subset V_{j+1} \subset \ldots H_0^r(\Omega)$

$$\mathrm{clos}_{H^r(\Omega)} \left(\bigcup_{j=j_0}^{\infty} V_j \right) = H_0^r(\Omega)$$

Linear orthogonal projectors $Q_j : H_0^r(\Omega) \to V_j$ s.th. $Q_j Q_\ell = Q_j$ for $j \leq \ell \rightsquigarrow Q_j - Q_{j-1}$ projector

Corollary

(S) Φ_j uniformly stable basis for V_j: $\quad \|\mathbf{c}\|_{\ell_2} \sim \|\mathbf{c}^T \Phi_j\|_{L_2(\Omega)}$

(J) Jackson estimate

$$\inf_{v_j \in V_j} \|v - v_j\|_{L_2(\Omega)} \lesssim 2^{-sj} \|v\|_{H^s(\Omega)} \qquad v \in H^s(\Omega) \qquad 0 < s \leq \delta$$

(B) Bernstein inequality $\qquad \|v_j\|_{H^s(\Omega)} \lesssim 2^{sj} \|v_j\|_{L_2(\Omega)} \qquad v_j \in V_j \qquad s < \tau$

$\Longrightarrow$ Norm equivalence

$$\|v\|_{H^r(\Omega)}^2 \sim \sum_{j=j_0}^{J} 2^{2rj} \|(Q_j - Q_{j-1})v\|_{L_2(\Omega)}^2 \qquad v \in V_J$$

Proof: Norm equivalence on $\hat{\Omega}$ [Dahmen, Kunoth '92] [Oswald '92]
(J) and discrete Hardy inequality $\rightsquigarrow$ upper estimate for $\|\cdot\|_{H^s(\Omega)}$
(B), $\|Q_j\|_{L_2(\Omega)} \lesssim 1$ and Whitney estimate $\rightsquigarrow$ lower estimate
F regular mapping

Norm Equivalence for Optimal Preconditioning

Corollary: For $H_0^r(\Omega)$ $\quad C_J^{-1} := A_{j_0} Q_{j_0} \circ \mathbf{F}^{-1} + \sum_{j=j_0}^{J} 2^{2rj}(Q_j - Q_{j-1}) \circ \mathbf{F}^{-1}$

is optimal preconditioner for $A_J : V_J \to V_J$: $\quad \mathrm{cond}_2(C_J^{1/2} A_J C_J^{1/2}) \sim 1 \quad$ as $J \to \infty$

Proof: [Jaffard '92], [Dahmen, Kunoth '92], [Oswald '92]
Isomorphism $\|Av\|_{H^{-r}(\Omega)} \sim \|v\|_{H^r(\Omega)}$ combined with norm equivalence for $H_0^r(\Omega)$ and $\mathbf{F}$ regular mapping

BPX realization of C_J^{-1}: replace $C_J = A_{j_0} Q_{j_0} \circ \mathbf{F}^{-1} + \sum_{j=j_0}^{J} 2^{-2rj}(Q_j - Q_{j-1}) \circ \mathbf{F}^{-1}$

by spectrally equivalent preconditioner

$$\hat{C}_J := A_{j_0} Q_{j_0} \circ \mathbf{F}^{-1} + \sum_{j=j_0}^{J} 2^{-2rj} Q_j \circ \mathbf{F}^{-1} \qquad \text{is optimal}$$

developed by [Bramble, Pasciak, Xu '90], optimality proved by [Dahmen, Kunoth '92], [Oswald '92]
Hierarchical basis preconditioner by [Yserentant '89] not optimal

$$\hat{C}_J := A_{j_0} Q_{j_0} \circ \mathbf{F}^{-1} + \sum_{j=j_0}^{J} 2^{-2rj} Q_j \circ \mathbf{F}^{-1} \text{ using } Q_j = \sum_{i \in I_j} (\,\cdot\,, B_{i,j} \circ \mathbf{F}^{-1})_{L_2(\Omega)}\, B_{i,j} \circ \mathbf{F}^{-1}$$

$$\rightsquigarrow$$

$$G_J = A_{j_0}^{-1} \sum_{i \in I_0} (\,\cdot\,, B_{i,j_0} \circ \mathbf{F}^{-1})_{L_2(\Omega)}\, B_{i,j_0} \circ \mathbf{F}^{-1}$$

$$+ \sum_{j=j_0+1}^{J} 2^{-2jr} \sum_{i \in I_j} (\,\cdot\,, B_{i,j} \circ \mathbf{F}^{-1})_{L_2(\Omega)}\, B_{i,j} \circ \mathbf{F}^{-1}$$

Implementation: refinement relation for B-splines

$\rightsquigarrow$ prolongation $\mathbf{I}_j^{j+1} : V_j \to V_{j+1}$ restriction $\mathbf{I}_{j+1}^{j} = (\mathbf{I}_j^{j+1})^T$

For $p = 2$, restriction is $\mathbf{I}_{j+1}^{j} = 2^{-1/2} \begin{bmatrix} \frac{1}{2} & \frac{9}{8} & \frac{3}{8} & & & & & & & \\ & \frac{1}{4} & \frac{3}{4} & \frac{3}{4} & \frac{1}{4} & & & & & \\ & & & \frac{1}{4} & \frac{3}{4} & \frac{3}{4} & \frac{1}{4} & & & \\ & & & & & \ddots & \ddots & & & \\ & & & & & \frac{1}{4} & \frac{3}{4} & \frac{3}{4} & \frac{1}{4} & \\ & & & & & & & \frac{3}{8} & \frac{9}{8} & \frac{1}{2} \end{bmatrix} \in \mathbb{R}^{2^j \times 2^{j+1}}$

$$\rightsquigarrow \quad \mathbf{I}_j^J := \mathbf{I}_{J-1}^J \mathbf{I}_{J-2}^{J-1} \cdots \mathbf{I}_j^{j+1} \quad \text{and} \quad \mathbf{I}_J^j := \mathbf{I}_{j+1}^j \mathbf{I}_{j+2}^{j+1} \cdots \mathbf{I}_J^{J-1}$$

$$\rightsquigarrow \qquad \tilde{\mathbf{G}}_J = \sum_{j=j_0}^{J} \mathbf{I}_j^J (\operatorname{diag} \mathbf{A}_j)^{-1} \mathbf{I}_J^j \circ \mathbf{F}^{-1}$$

First Numerical Results

Condition numbers $\mathrm{cond}_2(\tilde{\mathbf{G}}_J \mathbf{A}_J)$ for Laplacian on $\Omega = (0,1)^d$ for $d = 1, 2, 3$
with above BPX preconditioning

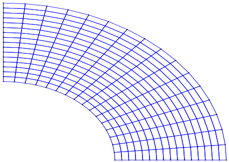
level j	interval				square				cube			
	$p=1$	$p=2$	$p=3$	$p=4$	$p=1$	$p=2$	$p=3$	$p=4$	$p=1$	$p=2$	$p=3$	$p=4$
3	7.43	3.81	7.03	5.93	5.93	7.31	22.8	133	3.49	39.5	356	5957
4	8.87	4.40	9.47	7.81	5.00	9.03	40.2	225	4.85	50.8	624	9478
5	10.2	4.67	11.0	9.36	5.70	9.72	51.8	293	5.75	56.6	795	11887
6	11.3	4.87	12.1	10.7	6.27	10.1	58.7	340	6.40	59.7	895	13185
7	12.2	5.00	12.7	11.5	6.74	10.4	63.1	371	6.91	61.3	961	13211
8	13.0	5.10	13.0	11.9	7.14	10.5	66.0	391	7.34	62.2	990	13234
9	13.7	5.17	13.2	12.1	7.48	10.6	68.0	403	7.70	62.6	1016	13255
10	14.2	5.22	13.4	12.2	7.77	10.6	69.3	411	7.99	62.9	1040	

$d = 1$: no dependence on p

Numerical Results: Dependence on Parametric Mapping **F**

Condition numbers of the BPX-preconditioned Laplacian on an analytic arc

level	$p = 1$	$p = 2$	$p = 3$	$p = 4$
3	5.04	12.4	31.8	184
	(21.8)	(8.64)	(31.8)	(184)
4	11.1	16.3	54.7	291
	(90.2)	(34.3)	(32.9)	(173)
5	25.3	19.0	70.1	376
	(368)	(139)	(98.9)	(171)
6	31.9	21.4	79.2	436
	(1492)	(560)	(401)	(322)
7	37.4	23.1	84.4	471
	(6015)	(2255)	(1620)	(1297)
8	42.1	24.3	87.3	490
	(241721)	(9062)	(6506)	(5217)
9	45.7	25.2	89.0	500
	(969301)	(36353)	(26121)	(20945)
10	48.8	25.9	90.1	505
	(388690)	(145774)	(104745)	(83975)

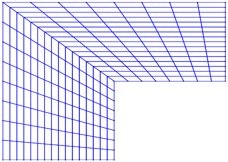


(numbers in brackets: no preconditioning)

Numerical Results: Dependence on Parametric Mapping **F**

…relative to a $\mathcal{C}^0$-parametrization of an L-shaped domain

level	$p = 1$	$p = 2$	$p = 3$	$p = 4$
3	14.0	13.4	33.5	194
	(25.8)	(10.2)	(33.5)	(194)
4	25.2	20.6	56.7	301
	(108)	(41.1)	(34.7)	(182)
5	36.9	26.8	72.1	383
	(452)	(168)	(123)	(180)
6	47.9	31.8	80.5	442
	(1845)	(689)	(500)	(400)
7	57.4	35.4	85.5	477
	(7465)	(2790)	(2025)	(1620)
8	65.3	38.0	88.3	496
	(30047)	(11244)	(8157)	(6533)
9	71.8	40.0	90.0	505
	(120603)	(45172)	(32773)	(26264)
10	77.0	41.2	91.0	511
	(483618)	(181140)	(131418)	(105381)

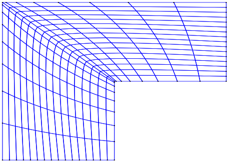


(numbers in brackets: no preconditioning)

Numerical Results: Dependence on Parametric Mapping **F**

Condition numbers of the BPX-preconditioned Laplacian relative to a singular $\mathcal{C}^1$-parametrization of an L-shaped domain

level	$p = 1$	$p = 2$	$p = 3$	$p = 4$
3	15.0	14.7	32.8	185
	(28.8)	(13.2)	(32.8)	(185)
4	44.1	36.2	56.7	303
	(133)	(53.7)	(38.5)	(189)
5	91.1	70.9	95.7	388
	(568)	(225)	(158)	(196)
6	167	147	155	463
	(2341)	(931)	(639)	(557)
7	443	385	385	887
	(9502)	(3804)	(2587)	(2350)
8	1136	960	1021	2417
	(38544)	(15353)	(10491)	(9604)
9	2797	2301	2588	6251
	(155276)	(61619)	(42355)	(38695)
10	6664	5362	6318	15505
	(622565)	(247091)	(169844)	(155143)



Further Improvement: BPX with SSOR

Split $\mathbf{A}_j = \mathbf{L}_j + \mathbf{D}_j + \mathbf{L}_j^T$ and replace $(\operatorname{diag} \mathbf{A}_j)^{-1}$ by SSOR preconditioner

$$\tilde{\mathbf{G}}_J = \sum_{j=j_0}^{J} \mathbf{I}_j^J (\mathbf{D}_j + \mathbf{L}_j)^{-T} \mathbf{D}_j (\mathbf{D}_j + \mathbf{L}_j)^{-1} \mathbf{I}_J^j \circ \mathbf{F}^{-1}$$

(application in optimal $\mathcal{O}(N)$ complexity)

Spectral condition numbers of BPX-preconditioned Laplacian for cubic B-splines ($p = 3$) on different geometries and using a SSOR preconditioning on each level

level	square, simple BPX	square	analytic arc	$\mathcal{C}^0$-map of L-shaped domain	singular $\mathcal{C}^1$-map of L-shaped domain
3	22.8	3.61	3.65	3.67	3.80
4	40.2	6.58	6.97	7.01	7.05
5	51.8	8.47	10.2	10.2	14.8
6	58.7	9.73	13.1	13.2	32.2
7	63.1	10.5	14.9	15.2	77.7
8	66.0	11.0	15.9	16.3	180
9	68.0	11.2	16.5	17.0	411
10	69.3	11.4	16.9	17.7	933

Ingredients for Efficient Numerical Solution: Nested Iteration

Recall goal: realize discretization error accuracy $\varepsilon_J \sim h^2 \sim 2^{-2J}$ for grid with spacing $h \sim 2^{-J}$ with minimal amount of work $\mathcal{O}(N)$ $N \sim 2^{Jd}$ unknowns

Naive strategy:

- Iterate only on highest level J and iterate until discretization error accuracy needs $\mathcal{O}(J) = \mathcal{O}(-\log \varepsilon_J)$ iterations to achieve prescribed discretization error accuracy $\varepsilon_J \sim 2^{-2J}$
- Each application of optimally conditioned $\mathbf{A}_J$ requires $\mathcal{O}(N_J)$ arithmetic operations
  $\rightsquigarrow$ a total of $\mathcal{O}(J\,N_J)$ arithmetic operations iterating on finest level only

Theorem:
Starting with coarsest level j_0, solve $\mathbf{A}_j \mathbf{y}_j = \mathbf{f}_j$ on each level j up to discretization error accuracy ε_j and prolongate result from level j to next level $j+1$ as initial guess

$\rightsquigarrow$ Optimal preconditioner + nested iteration yields method of optimal complexity $\mathcal{O}(N_J)$

to reach discretization error accuracy on finest level J

Proof: For mupliplicative Schwarz schemes: known as full multigrid
For additive preconditioners: optimal condition of $\mathbf{A}_j \rightsquigarrow$ fixed amount of iterations on each level to reach discretization error accuracy on that level;
spaces nested and $N_j \sim 2^{dj}$ and geometric series argument [Dahmen, Kunoth, Schneider '99]

Summary

- Proof for optimal multilevel preconditioner such that $\text{cond}_2(\tilde{\mathbf{G}}_J \mathbf{A}_J) = \mathcal{O}(1)$ based on generating system from hierarchical B-splines
  ($\neq$ (suboptimal) hierarchical basis preconditioner)
- Optimal complexity of the solver: optimal preconditioner + nested iteration
- Use of SSOR within BPX strongly reduces effect of isomorphism constants, parametric mappings. . .

Remarks and Outlook

- Multigrid preconditioners for high order FE based on low order functions [Sundar, Stadler, Biros '14]
- Extensions for nurbs, generalized splines
- Dependence of condition numbers on p ? [Manni et al '14], [Gahalaut, Tomar, Douglas '14], [Zulehner et al '14]
- A-posteriori error estimates $\rightsquigarrow$ local refinement Gianellli, Jüttler, Simeon, Vazquez . . .
  $\rightsquigarrow$ BPX preconditioner possible

(Further) Literature

BPX for C^0, C^1 elements [Dahmen, Kunoth '92] [Kunoth '94] [Oswald '92, '94]
BPX for C^0, C^1 elements on sphere [Maes, Kunoth, Bultheel '07]
(Monotone) multigrid for higher order B-splines (for pricing American options) [Holtz, Kunoth '07]

Iteration history for basis functions of degree $p = 2$ and $p = 3$