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Problem Set 07

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Problem 1: (5 points)
(a) Show that the skewness of a random variable having a symmetric pdf is zero;
(b) Show that the kurtosis of a Gaussian random variable is zero.

Problem 2: (5 points)
Show that a Gaussian variable has the largest entropy among all random variables.
Hint: consider a maximization problem of the differential entropy $H(f)$ over all probability densities f satisfying
(a) $f(x) \geq 0$, with equality outside the support set $\mathbb{S}$;
(b) $\int_{\mathbb{S}} f(x)dx = 1$;
(c) $\int_{\mathbb{S}} f(x)r_i(x)dx = \alpha_i$, for $1 \leq i \leq m$;
where f is a density on support set $\mathbb{S}$ meeting certain moment constraints $\alpha_1, \alpha_2, \ldots, \alpha_m$.

Problem 3: (5 points)
(a) Given a standardized Gaussian density

$$\varphi(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}}. \quad (1)$$

Define the Chebyshev-Hermite polynomials $h_i(\xi)$ by the derivatives of $\varphi(\xi)$

$$\frac{\partial^i \varphi(\xi)}{\partial \xi^i} = (-1)^i h_i(\xi)\varphi(\xi). \quad (2)$$

Compute the three first Chebyshev-Hermite polynomials $h_1(\xi), h_2(\xi)$ and $h_3(\xi)$;
(b) In fact, these polynomials have a nice orthonormal property

$$\int \varphi(\xi)h_i(\xi)h_j(\xi)d\xi = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases} \quad (3)$$

For a given non-Gaussian variable x, its pdf could be approximated by the following Gram-Charlier expansion truncated to first four terms

$$p_x(\xi) \approx \tilde{p}_x(\xi) = \varphi(\xi)\left(1 + \kappa_3(x)\frac{h_3(\xi)}{3!} + \kappa_4(x)\frac{h_4(\xi)}{4!}\right), \quad (4)$$

where $\kappa_3(x)$ and $\kappa_4(x)$ represent the third- and fourth-order cumulants, the skewness and kurtosis. Then the differential entropy $H(x)$ can be approximated by

$$H(x) \approx -\int \tilde{p}_x(\xi) \log \tilde{p}_x(\xi) d\xi. \quad (5)$$

Based on the orthonormal property (3), show that (5) can be further represented by

$$H(x) \approx -\int \varphi(\xi) \log \varphi(\xi) d\xi - \frac{\kappa_3(x)^2}{2 \times 3!} - \frac{\kappa_4(x)^2}{2 \times 4!}. \quad (6)$$

Problem 4: (5 points) Check the central limit theorem with computer
Let $x(t), t = 1, \ldots, T$, be independent random numbers distributed uniformly on the interval $[-1, 1]$, and

$$y_T = \sum_{t=1}^{T} x(t) \quad (7)$$

their sum. Generate 5000 different realizations of the random variable y_T at different value $T = 2, T = 4, T = 12$, and plot the corresponding histogram of y_T, and calculate the mean and variance of y_T.