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Multilevel preconditioning

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Summary. This paper is concerned with multilevel techniques for preconditioning linear systems arising from Galerkin methods for elliptic boundary value problems. A general estimate is derived which is based on the characterization of Besov spaces in terms of weighted sequence norms related to corresponding multilevel expansions. The result brings out clearly how the various ingredients of a typical multilevel setting affect the growth rate of the condition numbers. In particular, our analysis indicates how to realize even uniformly bounded condition numbers. For example, the general results are used to show that the Bramble-Pasciak-Xu preconditioner for piecewise linear finite elements gives rise to uniformly bounded condition numbers even when the refinements of the underlying triangulations are highly nonuniform. Furthermore, they are applied to a general multivariate setting of refinable shift-invariant spaces, in particular, covering those induced by various types of wavelets.

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1. Introduction

The perhaps most favorable principal advantages of finite element Galerkin methods for elliptic boundary value problems are the ability of coping with complicated domains and the fact that the existence of certain compactly supported bases, commonly referred to as *nodal bases*, gives rise to sparse stiffness matrices. However, one still has to deal with the fact that the (spectral) condition numbers of these matrices typically exhibit a polynomial growth rate with respect

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to their size. It is therefore of primary importance to *precondition* the corresponding linear systems, in particular, to exploit the full potential of conjugate gradient methods. Among the various approaches to this problem, the *hierarchical* preconditioner [Y1] as well as the closely related Bramble-Pasciak-Xu-preconditioner [BPX, Y2] have been attracting considerable attention over the past few years. In both cases the preconditioning is realized through a specific change of bases, which works when the finite element space under consideration results from successively refining some fixed (typically low-dimensional) initial space. The growthrate of the condition numbers for the corresponding hierarchical and BPX stiffness matrices was then shown to be only logarithmic in the size of the problem [Y1, BPX]. These results are formulated for second order problems in two spatial dimensions, i.e., for C^0-finite element spaces. Of course, the dramatic improvement offered by these methods initiated various attempts to extend these results to higher order problems and to higher spatial dimensions. However, already for three variables the hierarchical preconditioner produces polynomial growth. In general, it is fair to say that an extension of these results for finite elements with respect to higher dimensional simplicial partitions is a delicate problem. As for higher order problems in the 2*D*-case, logarithmic rates could recently be established for hierarchical and BPX type preconditioners based on C^1-conforming finite element spaces [O2, DOS].

On the other hand, the general setting for hierarchical and BPX bases exhibits striking analogies to concepts like *multiresolution analysis* and *wavelet expansions* [Mal, M]. The related rapidly expanding theory and applications have addressed mainly issues of harmonic analysis, image and sound processing and data compression. However, there have been various recent attempts to use wavelets also for the solution of partial differential equations (see e.g. [Awa, GLRT]), although mainly for problems in a single space variable. The particular issue of preconditioning linear systems arising from second order elliptic problems is also addressed in [J] but with respect to orthonormal wavelets of non-compact support whose construction is highly dependent on the domain. The same problem is treated in [CW], where it is shown that tensor products of compactly supported univariate orthonormal wavelets give rise to logarithmically growing condition numbers as obtained for the hierarchical bases. At any rate, these results do not seem to offer any essential advantages over the finite element based methods yet.

This motivates the present attempt to develop sufficiently general conditions for estimating condition numbers which, on one hand, apply in a finite element context and, on the other hand, allow to work out precisely the potential advantages of wavelet-type expansions, or better, of sequences of nested principal shift-invariant spaces. The essential task in all the above mentioned investigations is to relate Sobolev norms to certain discrete norms depending on the spaces to be used for the Galerkin approximations. We will establish such norm estimates for a general setup covering, in particular, bivariate C^0- and C^1-finite elements as well as sequences of nested shift-invariant spaces corresponding to any number of spatial dimensions and any order of differential operators. In fact, we will bring out the precise circumstances under which even uniformly bounded condition numbers are obtained. In particular, we will see that orthonormality is not crucial in this regard.

One should mention that some of the basic ingredients of our analysis are quite familiar facts which, however, come up in somewhat different areas. In particular, the theory of function spaces (cf. [N, T]) offers powerful tools for the present context. Specifically, we draw upon techniques used in [DP1], [DP2] and [DJP]. Similar techniques have been used in [O1, O2] for bivariate problems. After completing this paper, we also became aware of related work in [O3] dealing with a shift-invariant setting of arbitrary spatial dimension but only for certain C^0-finite elements.

Aside from the fact that sharp estimates are obtained for a general setup, including conventional finite elements, we would like to stress some particularly favorable features offered by the use of refinable shift-invariant spaces. Firstly, there is no need for developing explicit geometrical refinement strategies, which tend to become more complicated for higher spatial dimensions, since the different scales are introduced automatically through the translates and dilates of a single function, an essentially dimension independent concept. Secondly, the wavelet transform techniques, in particular the structure of so called *refinable* functions, offer a unified framework of powerful tools for dealing with basic tasks like computing inner products or generating the preconditioners. On the other hand, the elegance and generality of the techniques employed in this setting rests to some extent on the fact that in this case we have isolated the central issue of establishing the relevant norm estimates from the equally important problem of handling boundary conditions. However, we feel that the above mentioned advantages of a rather flexible and widely applicable concept justify investing separate systematic investigations of this issue which will be taken up in a forthcoming paper. Therefore, we will confine our discussion here to simple model cases, such as periodic or simple homogeneous boundary conditions to which the basic norm estimates apply without serious additional complications. We hope that clarifying those aspects of the problem, which relate to norm estimates, helps bringing those issues into focus which are crucial for further advancements.

The paper is organized as follows. In Sect. 2 we state the basic problem and collect some preliminaries. In Sect. 3 we formulate general conditions for estimating condition numbers of stiffness matrices for elliptic problems. These criteria are proved in Sect. 4 using techniques from the theory of function spaces. In particular, this leads to the characterization of the elements of Besov spaces in terms of weighted coefficient sequences of corresponding multilevel representations. In Sect. 5 we apply the general criteria to show that for second order problems and classical C^0-finite element spaces the Bramble-Pasciak-Xu preconditioner gives rise to uniformly bounded condition numbers even when the corresponding refined triangulations are highly *nonuniform*. Our primary objective, however, is to show in Sect. 6 that the preceding results apply to a rather general setting of *refinable*, *shift-invariant* spaces. The corresponding norm estimates are derived first relative to all of $\mathbb{R}^s$. We comment then on the case of bounded domains and simple types of boundary conditions. We focus on those cases where even uniformly bounded condition numbers are obtained. This includes wavelet type expansions, prewavelets and biorthogonal expansions. Again orthogonality of decompositions is *not* crucial in this context. Finally we comment briefly on computational issues related to implementing corresponding methods.

2. Preliminaries

Let us denote for any polynomial P on $\mathbb{R}^s$ by $P(D)$ the differential operator obtained by replacing each variable by the corresponding partial derivative. We will be interested in solving boundary value problems of the type

$$P(D)u = f \quad \text{on } \Omega, \qquad Bu = g \quad \text{on } \partial\Omega\,, \tag{2.1}$$

where P is a polynomial of degree $2k$, $\Omega \subset \mathbb{R}^s$ is some bounded domain with sufficiently regular boundary $\partial\Omega$, and B is to express suitable boundary conditions. Here and throughout the paper 'sufficiently regular' is to imply the following two things. Firstly, the standard L_p-moduli of smoothness are equivalent to corresponding K-functionals (cf. [DDS, JS]). Secondly, there exist appropriate extension operators from Besov spaces on Ω to Besov spaces on all of $\mathbb{R}^s$ (see [DS, JW]). So, in view of the interrelation of the various relevant regularity conditions on open domains established in [Sh], it is sufficient for our purposes to assume that Ω is *minimally smooth* in the sense of Stein [S]. Moreover, to us it is only important to assume that (2.1) is elliptic in the following sense. Suppose the weak formulation of (2.1) is to find $u \in H^k_B(\Omega)$ such that

$$a(u, v) = (f, v), \quad v \in H^k_B(\Omega)\,, \tag{2.2}$$

where $(u, v) := \int_\Omega u(x)v(x)dx$, $a(\cdot\,,\cdot)$ is the bilinear form induced by $(P(D), B)$, and $H^k_B(\Omega)$ is a suitable subspace of $H^k(\Omega)$ depending on the nature of the boundary conditions in terms of the operator B. We require that

$$a(\cdot\,,\cdot) \sim \|\cdot\|^2_{k,2}(\Omega)\,. \tag{2.3}$$

Here $F_1 \sim F_2$ will always mean that there exist some constants $0 < a, b < \infty$ such that $aF_1 \leqq F_2 \leqq bF_1$ holds independent of any parameters the quantities F_i may depend on and $\|\cdot\|_{k,p}(\Omega)$ is the Sobolev norm for the space $W^{k,p}(\Omega)$ (see e.g. [A]). We will adopt the standard convention of writing $H^k(\Omega) = W^{k,2}(\Omega)$ while otherwise p may range between 1 and infinity with the usual modification for $p = \infty$.

For any given finite dimensional subspace S of $H^k_B(\Omega)$, let A denote the positive definite selfadjoint operator on S defined by

$$a(u, v) = (Au, v), \quad v \in H^k_B(\Omega)\,. \tag{2.4}$$

Defining then $b \in S$ by $(b, v) = (f, v)$, $v \in S$, we have to solve the linear operator equation

$$Au = b \tag{2.5}$$

for some $u \in S$. Usually (2.5) is solved by means of an iteration of the form

$$u^{n+1} := (I - \omega CA)u^n + \omega Cb, \quad n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}\,, \tag{2.6}$$

or by a conjugate gradient acceleration of (2.6). Here ω is a suitable relaxation parameter and C is some selfadjoint positive definite operator approximating A^{-1}, and hence plays the role of a preconditioner. More precisely, to accomplish a favorable convergence of any such iteration scheme, C has to be chosen so as to keep the spectral condition number $\kappa(C^{1/2}AC^{1/2})$ as small as possible. To estimate condition numbers we will make use of the following well-known fact.

Remark 2.1. If for some constants $0 < \gamma, \Gamma < \infty$

$$\gamma(C^{-1}g, g) \leqq a(g, g) \leqq \Gamma(C^{-1}g, g), \quad g \in S, \tag{2.7}$$

then

$$\kappa(C^{1/2}AC^{1/2}) \leqq \Gamma/\gamma\ . \tag{2.8}$$

We conclude this section by addressing the role of specific realizations of the operators A and C. To this end, suppose $\Phi = \{\varphi_i : i \in I\}$ is a basis of S so that A may be represented in terms of the 'stiffness matrix' $A_\Phi := (a(\varphi_i, \varphi_j))_{i,j \in I}$. When Φ is a typical nodal basis of a classical space S of conforming finite elements in $H^k_B(\Omega)$, it is well-known that the spectral condition number $\kappa(A_\Phi)$ of A_Φ grows at least like $(\# I)^{2k/s}$. One way of viewing the task of preconditioning the system of linear equations

$$A_\Phi y = b_\Phi, \quad (b_\Phi)_i := (f, \varphi_i), \quad i \in I\ , \tag{2.9}$$

is to perform an appropriate change of basis. To this end, let $\Psi = \{\psi_i : i \in I\}$ be another basis of S, and let L denote the matrix that takes the coefficients of some $g \in S$ relative to Ψ into those relative to Φ. Thus the stiffness matrix relative to Ψ is then given by

$$A_\Psi = L^{\mathrm{T}} A_\Phi L\ . \tag{2.10}$$

Noting that for $g = \sum_{i \in I} y_i \psi_i$, one has $a(g, g) = y^{\mathrm{T}} A_\Psi y$, the equivalence (2.3) immediately implies the following fact which will be used later.

Remark 2.2. If for a given space S as above there exist some constants $0 < \gamma, \Gamma < \infty$ and $d_i, i \in I$, such that

$$\gamma \sum_{i \in I} |d_i y_i|^2 \leqq \|\sum_{i \in I} y_i \psi_i\|^2_{k,2(\Omega)} \leqq \Gamma \sum_{i \in I} |d_i y_i|^2, \quad y \in \mathbb{R}^{\# I}\ , \tag{2.11}$$

then there exists a constant c, depending only on the equivalence (2.3), such that

$$\kappa(D^{-1} A_\Psi D^{-1}) \leqq c\,\Gamma/\gamma\ ,$$

where $D = (d_i \delta_{i,l})_{i,l \in I}$.

Thus, when the quotient Γ/γ is small relative to $(\# I)^{2k/s}$, say, the matrix $C := (LD^{-1})(LD^{-1})^{\mathrm{T}}$ is a candidate for a preconditioner.

3. A general estimate

From a number of previous results one expects to obtain well-conditioned bases when the space S belongs to a sequence $\mathscr{S}$ of strictly *nested* spaces $S_j \subset H^k_B(\Omega), j \in \mathbb{N}_0$, i.e.,

$$S_0 \subset S_1 \subset S_2 \subset \dots,$$

where it is reasonable to require that the closure of their union with respect to the L_2-norm is all of $L_2(\Omega)$. A general multilevel framework may then be described by associating with $\mathscr{S}$ a sequence $\mathscr{Q}$ of *linear projectors* Q_j, $j \in \mathbb{N}_0$, which map any $S_m, m \geqq j$, onto S_j. Defining then

$$W_{j+1} = (Q_{j+1} - Q_j) S_{j+1} \tag{3.1}$$

yields the decomposition

(3.2)
$$S_{j+1} = S_j \oplus W_{j+1} .$$

It will be convenient to write $W_0 := S_0$ so that

$$S_m = \bigoplus_{j=0}^{m} W_j .$$

Clearly the corresponding *multilevel* representation of any $g \in S_m$ is then given by

(3.3)
$$g = Q_0 g + \sum_{j=1}^{m} (Q_j - Q_{j-1}) g .$$

To carry out our analysis, we introduce the difference operator

$$(\Delta_h^\ell f)(x) := \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^{\ell-j} f(x+jh), \quad x, h \in \mathbb{R}^s$$

and define for $1 \leqq p \leqq \infty$ the ℓ-th order L_p-modulus of smoothness

$$\omega_\ell(f, t, \Omega)_p := \sup_{|h|_2 \leqq t} \|\Delta_h^\ell f\|_p(\Omega_{\ell,h})$$

where $\Omega_{\ell,h} := \{x \in \Omega : x + jh \in \Omega, j = 0, \ldots, \ell\}$ and $|\cdot|_2$ denotes the Euclidean norm on $\mathbb{R}^s$. To see how the smoothness of f affects the decay rate of $\omega_\ell(f, t, \Omega)_p$, as t tends to zero, let us denote the usual Sobolev seminorms by $|f|^p_{\ell,p}(\Omega) := \sum_{|\alpha|=\ell} \|D^\alpha f\|^p_p(\Omega)$ for the whole range $1 \leqq p \leqq \infty$ of p with the usual modifications for $p = \infty$. Thus, when f belongs to the Sobolev space $W^{\ell,p}(\Omega)$, one can estimate $\omega_\ell(f, t, \Omega)_p$ in terms of $t^\ell |f|_{\ell,p}(\Omega)$. The latter quantity typically appears as an error bound for finite element approximations when the approximated function f is sufficiently smooth. When f is not known to be smooth beforehand, so called Whitney type estimates often still provide bounds for the approximation error in terms of the corresponding modulus of smoothness (see Sect. 4).

Our goal is to formulate a possibly unified yet sufficiently flexible framework of conditions, which yield good estimates for condition numbers for various types of finite elements as well as for a general multidimensional setting of nested shift-invariant spaces to be discussed later on. Since one expects, in particular, the norm and approximation properties of the projectors Q_j to come into play, we introduce, in view of the above remarks, the following quantity

(3.4)
$$v_{m,j} := \sup_{g \in S_m} \frac{\|Q_j g - g\|_2(\Omega)}{\omega_{k+1}(g, 2^{-j}, \Omega)_2} ,$$

and define

(3.5)
$$v_m := \max\{1, v_{m,j} : j = 0, \ldots, m\} .$$

We will formulate below in Theorem 3.1 a general estimate of condition numbers in terms of the quantity v_m. Thus, it remains to bound v_m. On one hand, this could be done directly for any concrete application. In fact, we will demonstrate this in Sect. 5 in connection with piecewise linear finite elements on *non-uniformly* refined triangulations. There we use the above mentioned local Whitney type estimates to bound the numerators of $v_{m,j}$ by a modulus of smoothness. On the other hand, in many cases the $v_{m,j}$ can be estimated by bounding the projectors Q_j and estimating the L_2-distance of $g \in S_m$ from S_j. This will lead to simpler criteria for uniformly bounded condition numbers given in Theorem 3.2.

To state these results, let A_m denote the operator defined by (2.4) for $S = S_m$ and, recalling the multilevel representation (3.3), let C_m^{-1} be the selfadjoint positive definite operator on S_m defined by

$$(3.6)\quad (C_m^{-1} g, g') = (Q_0 g, Q_0 g') + \sum_{j=1}^{m} 2^{2jk}((Q_j - Q_{j-1})g, (Q_j - Q_{j-1})g') .$$

We are now in a position to formulate the following general estimate.

Theorem 3.1. *Suppose $\mathscr{S}$ and $\mathscr{Q}$ have the following properties. There exists some real number $\gamma > k$ such that the Bernstein estimate*

$$(3.7)\qquad \omega_{k+1}(g, t, \Omega)_2 \leqq c(\min\{1, t2^m\})^\gamma \|g\|_2(\Omega), \quad g \in S_m ,$$

holds for some constant c independent of m and g. Then one has for C_m defined by (3.6)

$$(3.8)\qquad \kappa(C_m^{1/2} A_m C_m^{1/2}) = O((v_m)^2), \quad m \to \infty .$$

The Bernstein estimate (3.7) is known to hold for several classical finite element spaces (see e.g. [O1, O2]). In these cases as well as in the applications discussed in this paper it is verified by representing $g \in S_m$ in terms of a *stable* basis which reduces the problem to estimating the modulus of smoothness of the basis functions. One can usually exploit the differentiability properties of the basis functions to handle this latter task.

The proof of Theorem 3.1 and the following results will be postponed to the next section, where they will be derived from certain discrete norm estimates which can be formulated in a more general context. We proceed to state a consequence of Theorem 3.1 which provides a simple criterion for *uniformly bounded* condition numbers.

Theorem 3.2. *Suppose that the hypotheses of Theorem* 3.1 *are satisfied and that $\mathscr{Q}$ is uniformly bounded on $L_2(\Omega)$. Moreover, assume that there exists an integer $\ell > k$ such that $\mathscr{S}$ satisfies the Jackson estimate*

$$(3.9)\qquad \inf_{g \in S_m} \|f - g\|_2(\Omega) \leqq c 2^{-m\ell} |f|_{\ell,2}(\Omega), \quad f \in H^\ell(\Omega) ,$$

for some constant c independent of m and f. Then

$$\kappa(C_m^{1/2} A_m C_m^{1/2}) = O(1), \quad m \to \infty .$$

Typical examples to which Theorem 3.2 immediately applies are C^0-piecewise linear finite elements on uniformly refined triangulations in the context of second order bivariate elliptic problems (cf. [BPX, Y1]), or the C^1-conforming finite elements for fourth order problems constructed in [O2] and [DOS]. In all these cases the Bernstein and Jackson estimates (3.7) and (3.9) are easily verified. Thus the corresponding versions of the Bramble-Pasciak-Xu (BPX) preconditioner, where Q_j is the orthogonal projector onto S_j, are readily seen to yield uniformly bounded condition numbers (see also [O3]). However, when dealing with *nonuniform* refinements of triangulations in the sense of [BSW], the Jackson estimate (3.9) cannot be expected to hold in this form so that Theorem 3.2 is not applicable. We will address this issue in Sect. 5 and show that Theorem 3.1 still applies without much additional effort.

The actual numerical realization of the BPX preconditioner in a finite element setting is not based on an explicit change of basis (see e.g. [Y2]). However, in Sect. 6 we shall encounter a wide class of cases where the construction of the preconditioners is conveniently based on a change of basis so that Remark 2.2 applies. We will therefore conclude this section by briefly reinterpreting the above results in terms of the corresponding stiffness matrices.

To this end, suppose that each W_j possesses a *stable* basis $\Psi_j := \{\psi_{j,i} : i \in I_j\}$, i.e., there exist constants $c_j, j \in \mathbb{N}_0$, such that

$$\left\| \sum_{i \in I_j} y_i \psi_{j,i} \right\|_2^2 (\Omega) \sim c_j^2 \sum_{i \in I_j} |y_i|^2, \quad y \in \mathbb{R}^{\# I_j} . \tag{3.10}$$

Moreover, let D denote the diagonal matrix defined by $D_{(j,i),(j',i')} = c_j 2^{kj} \delta_{j,j'} \delta_{i,i'}$, $j, j' = 0, \ldots, m$, $i, i' \in I_j$, and let M_m denote the stiffness matrix relative to the *multilevel basis* $\bigcup_{0 \leq j \leq m} \Psi_j$.

Corollary 3.1. *If in addition to the hypotheses in Theorem* 3.1, *respectively Theorem* 3.2, *condition* (3.10) *holds, then one has*

$$\kappa(D^{-1} M_m D^{-1}) = O((v_m)^2), \quad m \to \infty \quad ,$$

and

$$\kappa(D^{-1} M_m D^{-1}) = O(1), \quad m \to \infty \quad ,$$

respectively.

Finally, note that all the results stated here remain true when the scaling factor two is replaced by any other factor $\rho > 1$, say. Since in all the subsequent examples we will always consider the case $\rho = 2$, we dispense here with this formal generalization.

4. Discrete norm estimates

The proofs of Theorem 3.1 and Theorem 3.2 are based on techniques from the theory of function spaces (cf. e.g. [DJP, DP1, N, O1, T]). Similar arguments are used in [O2] in a more specific bivariate finite element setting. Although the above problem formulation requires only L_2-estimates, it takes hardly more effort to work in the general scale of L_p-spaces for $1 \leqq p \leqq \infty$ (with the usual modifications for $p = \infty$). In fact, one could consider quasi-norms including values p less than one (cf. e.g. [DP1]) which we will, however, dispense with here. We will adopt the standard notation for L_p- and Sobolev-norms and use $\|\lambda\|_{\ell_p} := (\sum_{j \in I} |\lambda_j|^p)^{1/p}$ to denote the corresponding sequence or vector norms. Since the domain Ω will be fixed throughout, we will drop any reference to Ω in our notation for the norms. In fact, the following reasoning works for $\Omega = \mathbb{R}^s$ and remains valid for any domain Ω satisfying the assumptions stated in Sect. 2.

Setting $Q_{-1} := 0$, the decomposition (3.3) suggests introducing the norms

$$\|g\|_{(r,m,p,q)} := \|\{2^{rj}\|(Q_j - Q_{j-1})g\|_p\}_{j=0}^m\|_{\ell_q} \tag{4.1}$$

on S_m, where we will always assume $r > 0$ and $1 \leqq p, q \leqq \infty$. In view of (2.11), our objective is to relate these norms to the Sobolev norms $\|\cdot\|_{k,2}$. In this context it is, however, useful to consider first the Besov spaces $B^r_{p,q}$ consisting of all functions in L_p such that

$$\|f\|_{B^r_{p,q}} := \|f\|_p + |f|_{B^r_{p,q}} < \infty,$$

where

$$|f|_{B^r_{p,q}} := \|\{2^{rj}\omega_\ell(f, 2^{-j})_p\}_{j \in \mathbb{N}_0}\|_{\ell_q},$$

and ℓ is any (fixed) integer larger than r (see e.g. [DP1]). Moreover, let

$$E_m(f)_p := \inf_{g \in S_m} \|f - g\|_p$$

denote the error of best L_p-approximation from S_m. To formulate the following result, let the numbers $v^{(p)}_{m,j}$ and $v^{(p)}_m$ be defined as in (3.4) and (3.5) but relative to L_p.

Theorem 4.1. *Suppose that for some real number $\gamma > k$,*

$$\omega_{k+1}(g, t)_p \leqq c(\min\{1, t2^m\})^\gamma \|g\|_p, \quad g \in S_m, \tag{4.2}$$

where c is independent of g and m. Then for each $0 < r < \min\{\gamma, k+1\}$, there exist constants $0 < c_1, c_2 < \infty$ independent of $g \in S_m, m \in \mathbb{N}_0$, such that

$$\frac{c_1}{v^{(p)}_m}\|g\|_{(r,m,p,q)} \leqq \|g\|_{B^r_{p,q}} \leqq c_2\|g\|_{(r,m,p,q)}, \quad g \in S_m. \tag{4.3}$$

Proof. Note that for $g \in S_m$,

$$\|g\|_{(r,m,p,q)} \leqq c(\|g\|_p + \|\{2^{jr}\|Q_j g - g\|_p\}_{j=0}^m\|_{\ell_q}). \tag{4.4}$$

By definition of $v^{(p)}_m$, we have for $j = 0, \ldots, m$,

$$\|Q_j g - g\|_p \leqq v^{(p)}_m \omega_{k+1}(g, 2^{-j})_p, \quad g \in S_m, \tag{4.5}$$

so that we conclude from (4.4)

$$(v^{(p)}_m)^{-1}\|g\|_{(r,m,p,q)} \leqq c\|g\|_{B^r_{p,q}}, \quad g \in S_m, \tag{4.6}$$

where c is independent of g and m. On the other hand, suppose

$$g = \sum_{j=0}^\infty g_j, \quad g_j \in S_j, \ j \in \mathbb{N}_0, \tag{4.7}$$

so that (4.2) yields

$$\begin{aligned}\omega_{k+1}(g, 2^{-n})_p &\leqq c\sum_{j=0}^\infty \omega_{k+1}(g_j, 2^{-n})_p \\ &\leqq c\left(\sum_{j=0}^n 2^{(j-n)\gamma}\|g_j\|_p + \sum_{j=n+1}^\infty \|g_j\|_p\right).\end{aligned}$$

Hence

$$|g|_{B^r_{p,q}} \leqq c\left(\left\| \left\{ 2^{n(r-\gamma)} \sum_{j=0}^{n} 2^{j\gamma} \|g_j\|_p \right\}_{n \in \mathbb{N}_0} \right\|_{\ell_q} + \left\| \left\{ 2^{nr} \sum_{j=n+1}^{\infty} \|g_j\|_p \right\}_{n \in \mathbb{N}_0} \right\|_{\ell_q} \right)$$

$$\leqq c \| \{2^{nr} \|g_n\|_p\}_{n \in \mathbb{N}_0} \|_{\ell_q} ,$$

where we have used a discrete Hardy-type inequality in the last step which is valid whenever $r < \gamma$ (see e.g. (5.2) in [DP1]). Specifically, choosing $g_n := (Q_n - Q_{n-1})g \in S_n$ provides

$$\|g\|_{B^r_{p,q}} \leqq c \|g\|_{(r,m,p,q)}, \quad g \in S_m , \tag{4.8}$$

where c is independent of g and m. Combining (4.6) and (4.8) proves the assertion of Theorem 4.1. □

We are now ready to prove some of the claims stated in Sect. 3.

Proof of Theorem 3.1. Fixing $p = q = 2, r = k$ we note that, by (4.1),

$$(C_m^{-1} g, g) = \sum_{j=0}^{m} 2^{2jk} \|(Q_j - Q_{j-1})g\|_2^2 = \|g\|^2_{(k,m,2,2)} , \tag{4.9}$$

where the operator C_m is defined by (3.6). Thus, Theorem 4.1 assures that

$$\frac{c_1}{v_m} (C_m^{-1} g, g)^{1/2} \leqq \|g\|_{B^k_{2,2}} \leqq c_2 (C_m^{-1} g, g)^{1/2}, \quad g \in S_m .$$

The assertion of Theorem 3.1 follows now from from Remark 2.1, and the fact that

$$W^{k,2} = B^k_{2,2}, \quad \|\cdot\|_{k,2} \sim \|\cdot\|_{B^k_{2,2}} \tag{4.10}$$

(see e.g. [T]). □

Proof of Corollary 3.1. Expanding $(Q_j - Q_{j-1})g$ in terms of the basis Ψ_j of W_j, i.e.,

$$(Q_j - Q_{j-1})g = \sum_{i \in I_j} \lambda_{j,i}(g) \psi_{j,i} ,$$

the stability condition (3.10) yields

$$\sum_{j=0}^{m} 2^{2jk} \|(Q_j - Q_{j-1})g\|_2^2 \sim \sum_{j=0}^{m} 2^{2jk} c_j^2 \sum_{i \in I_j} |\lambda_{j,i}(g)|^2, \quad m \in \mathbb{N}_0 .$$

Using again (4.9), this latter relation may be rewritten as

$$\|g\|^2_{(k,m,2,2)} \sim (D\lambda(g), D\lambda(g)) ,$$

where $\lambda(g) = (\lambda_{j,i}(g) : i \in I_j, j = 0, \ldots, m)$ and D is the diagonal matrix appearing in Corollary 3.1. Hence, applying again Theorem 4.1 with $p = q = 2, r = k < \gamma$, and taking (4.10) into account, we obtain

$$\frac{c_1}{v_m^2} (D\lambda(g), D\lambda(g)) \leqq \|g\|^2_{k,2} \leqq c_2 (D\lambda(g), D\lambda(g)) ,$$

where the constants c_1, c_2 are independent of $m \in \mathbb{N}_0$. The first assertion in Corollary 3.1 follows now from Remark 2.2. The second part of Corollary 3.1 is an immediate consequence of the first part combined with the following observation. □

Proposition 4.1. *If $\mathcal{Q}$ is a uniformly bounded sequence of projectors on L_p and if for some integer $\ell > k$,*

$$E_m(f)_p \leqq c 2^{-m\ell} |f|_{\ell,p}, \quad f \in W^{\ell,p}, \tag{4.11}$$

holds with some constant c independent of f and m, one has

$$v_m^{(p)} = O(1), \quad m \to \infty . \tag{4.12}$$

Proof. Following standard lines, we first note that for $v \in W^{\ell,p}$ we have, in view of (4.11),

$$\begin{aligned} E_m(f)_p &\leqq \|f - v\|_p + E_m(v)_p \\ &\leqq c(\|f - v\|_p + 2^{-m\ell}|v|_{\ell,p}) . \end{aligned} \tag{4.13}$$

Thus defining the K-functional

$$K_\ell(f, t, \Omega)_p = K_\ell(f, t)_p := \inf_{v \in W^{\ell,p}} \{\|f - v\|_p + t^\ell |v|_{\ell,p}\}$$

and recalling from [DDS, JS] that, under our assumptions on Ω,

$$K_\ell(f, t)_p \sim \omega_\ell(f, t)_p, \quad t \to 0 , \tag{4.14}$$

we readily conclude that for $f \in L_p$

$$E_m(f)_p \leqq c\, \omega_\ell(f, 2^{-m})_p, \quad m \in \mathbb{N}_0 . \tag{4.15}$$

Now the classical Lebesgue estimate yields

$$\|Q_j g - g\|_p \leqq (1 + \|Q_j\|_p) E_j(g)_p$$

which, in view of the definition of $v_m^{(p)}$, (4.15), and the fact that $\omega_\ell(\cdot, t)_p \leqq c\,\omega_{k+1}(\cdot, t)_p$ holds for some constant c depending on $\ell > k$, proves our claim. □

Proof of Theorem 3.2. The assertion of Theorem 3.2 is now an immediate consequence of Theorem 3.1 and Proposition 4.1 specialized to $p = 2$. □

As an aside, Theorem 4.1 provides the following characterization of Besov spaces.

Corollary 4.1. *Suppose that for $\mathcal{S}$ and $\mathcal{Q}$ one has in addition to the assumption* (4.2) *that $v_m^{(p)} = O(1), m \to \infty$. Then*

$$\|\cdot\|_{(r,\infty,p,q)} \sim \|\cdot\|_{B^r_{p,q}} \quad \text{for } 0 < r < \gamma .$$

Moreover, if (3.10) *holds for some p in the range $1 \leqq p \leqq \infty$ with $c_j = O(1)$, as $j \to \infty$, then $f = \sum_{j=0}^{\infty} \sum_{i \in I_j} \lambda_{j,i}(f) \psi_{j,i} \in B^r_{p,q}$ if and only if the sequence $\{2^{rj} \|\lambda_j(f)\|_{\ell_p}\}_{j \in \mathbb{N}_0}$ belongs to ℓ_q.*

5. The BPX scheme and nonuniform refinements

In this section we wish to apply the above general results to the classical case of second order elliptic problems in the plane although what will be said remains valid for three spatial variables as well. To be more specific we adhere precisely to the situation considered in [Y2], i.e., Ω is a bounded polygonal domain in $\mathbb{R}^2$, Dirichlet conditions are prescribed on some part Γ of its boundary $\partial\Omega$, and the differential operator $P(D)$ in (2.1) has the form $P(D) = -D^{\mathrm{T}}AD$ where $A = A(x)$ is a 2×2 positive definite matrix throughout Ω. In this case $\mathscr{S}$ can be taken as a sequence of C^0 piecewise linear finite element spaces relative to a sequence $\mathscr{T} = \{T_j\}_{j\in\mathbb{N}_0}$ of nested triangulations of Ω. By this we mean that T_{j+1} is obtained by subdividing the triangles in T_j in an appropriate fashion. Here a triangulation T of Ω is a collection of triangles with pairwise disjoint interiors such that

(T1) $$\Omega = \bigcup_{\tau\in T} \tau\,,$$

(T2) The intersection of any two triangles $\tau, \tau' \in T$ is either empty or a common vertex or a common edge.

It is important in this context that the angles of the resulting triangles remain bounded away from zero. This can be, for instance, achieved by subdividing *each* triangle in T_j into four congruent subtriangles, taking midpoints of edges as new vertices. This gives rise to *uniform* refinements of T_0, i.e. diam $\tau \sim 2^{-j}$ for $\tau \in T_j$. Recall that the BPX preconditioner [BPX] corresponds to choosing Q_j as orthogonal projection (relative to a possibly weighted inner product, see e.g. [Y2, B]) onto S_j. As pointed out in Sect. 3, in the case of uniform refinements, this is known to yield uniformly bounded condition numbers [O3].

However, realistic problems typically require locally refined grids, i.e., highly non-uniform triangulations, and the question arises whether the favorable behavior for uniform refinements persists. In [Y2], the bound $O(j^2)$ is established for the corresponding BPX scheme. Recently the better bound $O(j)$ has been proved in [B] for the same class of nonuniform refinements. In this section we will show that the condition numbers even remain *uniformly bounded* as in the case of uniform refinements.

We briefly recall the main rules for forming possibly nonuniform successive refinements of triangulations. More detailed discussions of these rules and their consequences can be found, for instance, in [B, BSW, DLY, L] and [Y2]. A triangle in T_j is either a triangle in T_{j+1}, or it is decomposed into four congruent subtriangles or into two subtriangles by connecting the midpoint of an edge with the opposite vertex. The first refinement is called *regular*, the second *irregular*. Accordingly, the triangles resulting from these refinements are called *regular* and *irregular*, respectively. Moreover, all triangles in the initial triangulation T_0 are regular.

Regular refinements usually result from some adaptive criterion. If a triangle τ is subjected to a regular refinement, according to such a criterion, while its neighbor τ', say, sharing an edge with τ, is not, the resulting partition would violate condition (T2) above. This is then remedied by an irregular refinement of τ'.

To ensure that interior angles remain bounded away from zero, one requires the following additional conditions (see e.g. [B, Y2]).

(R1) Irregular triangles are not refined further.

(R2) Only triangles $\tau \in T_j$ with $L(\tau) = j$ are refined for the construction of T_{j+1}, where

$$L(\tau) = \min\{i : \tau \in T_i\}$$

denotes the *level* of τ.

These constraints still admit highly nonuniform meshes. Nevertheless, the variation of diameters of nearby triangles turns out to be sufficiently gradual in the following sense.

Lemma 5.1. *Let the sequence of triangulations $\mathscr{T}$ be obtained by the above rules. Then there exists a finite positive constant c, depending only on T_0, such that*

$$\frac{\operatorname{diam} \tau}{\operatorname{diam} \tau'} \leqq c \quad \textit{for all } \tau, \tau' \in T_j, \quad \tau \cap \tau' \neq \emptyset \ . \tag{5.1}$$

Proof. Since this fact is important for our subsequent analysis we sketch a proof. It follows immediately from (R1), (R2) that every triangle occurring in any T_j is geometrically similar to some triangle in T_0 or to a triangle arising from an irregular refinement of some triangle in T_0, and hence to some element of a fixed finite collection of triangles. This has the following consequences: firstly,

$$\operatorname{diam} \tau \sim 2^{-L(\tau)} \ , \tag{5.2}$$

secondly, there exists an integer M, depending only on T_0 such that for any vertex v in any member T_j of $\mathscr{T}$

$$\#\{\tau \in T_j : v \in \tau\} \leqq M \ , \tag{5.3}$$

i.e., the number of triangles in any of the triangulations T_j sharing a common vertex is bounded independently of $j \in \mathbb{N}_0$.

Now it follows from (5.2) that it suffices to prove the existence of an integer K, depending only on T_0, such that for all $j \in \mathbb{N}_0$

$$|L(\tau) - L(\tau')| \leqq K \quad \text{for all } \tau, \tau' \in T_j, \quad \tau \cap \tau' \neq \emptyset \ . \tag{5.4}$$

To this end, suppose τ, τ' are any two triangles in T_j sharing a common edge η. We claim that

$$|L(\tau) - L(\tau')| \leqq 1 \ . \tag{5.5}$$

If $L(\tau) = L(\tau')$ there is nothing to show. So assume that $L(\tau') > L(\tau)$. Hence, denoting by τ'' the 'parent' of τ', (R2) assures that τ and τ'' both belong to some triangulation $T_{j'}$ where $j' < j$. By (T2) η must still be the common edge of τ and τ''. Therefore the refinement that generated τ' must have been irregular, i.e., τ' is an irregular triangle. By definition, one has $L(\tau'') = L(\tau') - 1$. But now we must have $L(\tau'') = L(\tau)$, because if we still had $L(\tau'') > L(\tau)$ the above reasoning would imply that also τ'' is irregular, contradicting (R2). This proves (5.5). Taking (5.3) into account, we immediately conclude now that (5.4) holds with $K = M$. This completes the proof of Lemma 5.1. □

Note that the corresponding spaces S_j are not just subspaces of those that would be obtained for uniform refinements. Consequently, the results for the uniform case do not apply directly.

To keep the estimates independent of the quasi-uniformity of the initial triangulation T_0, appropriate weighted L_2-norms are introduced in [Y2]. Since this is not essential for the question of boundedness, we dispense with this here to keep the exposition as simple as possible.

Let X_j denote the set of vertices in T_j except those belonging to Γ. Hence S_j is spanned by the piecewise linear 'hat' functions defined by

$$\varphi_{j,v}(w) = 2^{L_{j,v}}\delta_{vw}, \quad v, w \in X_j \,, \tag{5.6}$$

where $L_{j,v} := \min\{L(\tau) : \tau \in T_j,\ v \in \tau\}$. One easily confirms that

$$\int_\Omega \varphi_{j,v}(x)^2\,dx = 2^{2L_{j,v}} \mathrm{vol}_2(\operatorname{supp}\varphi_{j,v})/6$$

so that, in view of (5.1)

$$\|\varphi_{j,v}\|_2(\Omega) \sim 1, \quad v \in X_j, \quad j \in \mathbb{N}_0 \,. \tag{5.7}$$

Moreover, for $g = \sum_{v \in X_j} c_v \varphi_{j,v} \in S_j$ and any $\tau \in T_j$, one has

$$\|g\|_2^2(\tau) \leqq c \sum_{v \in X_{j,\tau}} |c_v|^2 \|\varphi_{j,v}\|_2^2(\Omega) \tag{5.8}$$

where the constant c depends only on the cardinality of the set $X_{j,\tau} := \{v \in X_j : v \in \tau\}$ which, in turn, is uniformly bounded in $\tau \in T_j$ and $j \in \mathbb{N}_0$. Conversely, taking the normalization (5.6) into account, and using a standard inverse estimate combined with (5.2) yields

$$|c_v|^2 = 2^{-2L_{j,v}}|g(v)|^2 \leqq c\, 2^{-2L_{j,v}} 2^{-2L_{j,v}} \|g\|_2^2(\tau) \tag{5.9}$$

where $L(\tau) = L_{j,v}$. Thus, upon summing over $\tau \in T_j$ in (5.8) and (5.9), we obtain by (5.7)

$$\Big\| \sum_v c_v \varphi_{j,v} \Big\|_2 (\Omega) \sim \|\{c_v\}_{v \in X_j}\|_{\ell_2} \,. \tag{5.10}$$

It is now straightforward to conclude from (5.10) (see e.g. [O1]) that

$$\omega_2(g, t, \Omega)_2 \leqq c(\min\{1, t2^j\})^{3/2} \|g\|_2(\Omega) \tag{5.11}$$

holds for all $g \in S_j$ where c is independent of j.

Note that, due to the fact that T_j may involve triangles of highly varying size, we cannot expect that a Jackson estimate of type (3.9) holds. Thus, in order to apply Theorem 3.1 or Theorem 4.1 we have to estimate the quantities v_m directly. This estimate will be based on certain auxiliary projectors which we introduce next. To this end, note that for any triangle τ in T_j with vertices u, v, w, say, the restrictions of $\varphi_{j,v'}$, $v' \in \{u, v, w\}$, to τ are clearly linearly independent over τ. Hence, there exists a unique collection of linear polynomials $\zeta_u^\tau, \zeta_v^\tau, \zeta_w^\tau$ such that

$$\int_\tau \varphi_{j,v'}(x)\zeta_{v''}^\tau(x)\,dx = \delta_{v'v''}, \quad v', v'' \in \{u, v, w\} \,. \tag{5.12}$$

Define for every $v \in X_j$, and $\tau \in T_j$ such that $v \in \tau$

(5.13)
$$\eta_{j,v}(x) = \begin{cases} \dfrac{1}{N_v}\zeta_v^\tau(x), & x \in \tau \\ 0, & x \notin \operatorname{supp} \varphi_{j,v} \end{cases},$$

where N_v is the number of triangles in T_j contained in the support of $\varphi_{j,v}$. Clearly, (5.12) and (5.13) imply that

(5.14)
$$(\eta_{j,w}, \varphi_{j,v}) := \int_\Omega \eta_{j,w}(x)\varphi_{j,v}(x)\,dx = \delta_{vw}, \quad v, w \in X_j,$$

so that

(5.15)
$$(\tilde{Q}_j f)(x) := \sum_{v \in X_j} (f, \eta_{j,v})\varphi_{j,v}(x)$$

defines, in contrast to the quasi-interpolants employed in [B, Y2], a *projector* onto S_j. Consequently, it not only reproduces locally constant functions but all elements of $\prod_1(\mathbb{R}^2)$, the set of polynomials of degree at most one on $\mathbb{R}^2$. Moreover, since all triangles occurring in the triangulations T_j are similar to triangles in T_0 or to triangles resulting from irregular refinements of T_0, one easily deduces from (5.7) and (5.12) that

(5.16)
$$\|\eta_{j,v}\|_2(\Omega) \sim 1, \quad v \in X_j, \quad j \in \mathbb{N}_0.$$

Thus, setting $\Omega_{j,\tau} = \bigcup\{\tau' \in T_j : \tau' \cap \tau \neq \emptyset\}$, we conclude from (5.7) and (5.16) that

(5.17)
$$\|\tilde{Q}_j f\|_2(\tau) = \Big\| \sum_{v \in X_{j,\tau}} (f, \eta_{j,v})\varphi_{j,v} \Big\|_2(\tau) \leqq c\|f\|_2(\Omega_{j,\tau}),$$

where c depends only on the initial triangulation T_0.

Let

(5.18)
$$T_j^* := \{\tau \in T_j : L(\tau) < j,\ \Omega_{j,\tau} \cap \tau' = \emptyset \text{ for all } \tau' \in T_j \text{ with } L(\tau') = j\},$$

and let g be any element in S_m. By the refinement rules (R1), (R2), all triangles in $T_j^*, j \leqq m$, belong also to T_m. Moreover, due to the localness of the supports of the dual basis functions $\eta_{j,v}$, and since $\tilde{Q}_j$ is a projector, we have

(5.19)
$$\|\tilde{Q}_j g - g\|_2(\tau) = 0, \quad \tau \in T_j^*.$$

For $\tau \in T_j \backslash T_j^*$, we obtain, in view of (5.17), for every $P \in \prod_1$

(5.20)
$$\|\tilde{Q}_j g - g\|_2(\tau) \leqq \|\tilde{Q}_j(g - P)\|_2(\tau) + \|g - P\|_2(\tau) \leqq c\|g - P\|_2(\Omega_{j,\tau}).$$

At this point we employ the following Whitney-type estimate for best local polynomial approximation which we state in its general form for L_p-norms and for polynomials on $\mathbb{R}^s$. To this end, we need the modified modulus of continuity

$$w_\ell(f, t, \Omega)_p := \left(t^{-s} \int_{[-t,t]^s} \|\Delta_h^\ell f\|_p^p(\Omega_{\ell,h})\,dh \right)^{1/p}.$$

Now let σ be any simplex in $\mathbb{R}^s$ and let $t = \operatorname{diam} \sigma$. The reasoning in [SO] (see also Lemma 3.1 in [DP1] or [O1]) can be used to show that

$$\inf_{P \in \prod_{\ell-1}(\mathbb{R}^s)} \| f - P \|_p(\sigma) \leqq c\, w_\ell(f, t, \sigma)_p \tag{5.21}$$

where the constant c depends only on the smallest angle in σ but not on f and t. One readily infers from (5.1) that

$$\operatorname{diam}(\Omega_{j,\tau}) \sim 2^{-j} \quad \text{for } \tau \in T_j \backslash T_j^* . \tag{5.22}$$

Since $P \in \prod_1(\mathbb{R}^2)$ was arbitrary, (5.21) assures, in view of (5.20), that

$$\| \tilde{Q}_j g - g \|_2(\tau) \leqq c\, w_2(g, 2^{-j}, \Omega_{j,\tau})_2 , \tag{5.23}$$

where c depends only on the shape of the triangles in the initial triangulation. Taking (5.19) into account and summing over $\tau \in T_j \backslash T_j^*$ yields

$$\| \tilde{Q}_j g - g \|_2(\Omega) \leqq c\, w_2(g, 2^{-j}, \Omega)_2 . \tag{5.24}$$

Again the arguments in [PP] carry over to the multivariate case to confirm that there exist constants $0 < c_1, c_2 < \infty$, depending only on ℓ, p and s, such that

$$c_1 w_\ell(f, t, \Omega)_p \leqq \omega_\ell(f, t, \Omega)_p \leqq c_2 w_\ell(f, t, \Omega)_p , \tag{5.25}$$

which yields

$$\| \tilde{Q}_j g - g \|_2(\Omega) \leqq c\, \omega_2(g, 2^{-j}, \Omega)_2 . \tag{5.26}$$

Since one trivially has

$$\| Q_j g - g \|_2(\Omega) \leqq \| \tilde{Q}_j g - g \|_2(\Omega) ,$$

we conclude from (5.26) that

$$v_m = O(1), \quad m \to \infty . \tag{5.27}$$

In view of (5.11) and (5.27), Theorem 3.1 or Theorem 4.1 therefore yields the following result.

Theorem 5.1. *Let $\mathcal{T}$ satisfy the conditions $(R1)$ and $(R2)$ above. The corresponding BPX preconditioner gives rise to uniformly bounded condition numbers.*

For the actual numerical realization of the BPX scheme, the reader is referred to the detailed discussion in [Y2].

Incidentally, we have shown that, on account of (5.26) the projectors $\tilde{Q}_j$ themselves induce efficient preconditioners.

In principle, one can follow similar lines to show that the BPX scheme still gives rise to uniformly bounded condition numbers in connection with nonuniform refinements of certain C^1 conforming finite element spaces for fourth order problems discussed in [DOS]. However, since this requires a bit more technical elaboration, it will be reported on elsewhere.

Finally, note that the construction of the projectors $\tilde{Q}_j$ works equally well for tetrahedralizations in the three dimensional case. Since the above Whitney-type estimates remain valid for any number of spatial variables and since analogous local refinements for tetrahedral partitions are available as well [Bä], the corresponding assertion of Theorem 5.1 still holds in the three dimensional case.

6. Refinable shift-invariant spaces

A central objective of this paper is to apply the above criteria to a possibly general multidimensional setting of refinable shift-invariant spaces generated by the dilates and integer translates of a single function φ on $\mathbb{R}^s$. Such a setting allows the realization of (local) refinements and multilevel 'zooming-in' techniques without worrying about explicit geometric refinement strategies for given domain partitions which are usually highly dimension dependent. Such a setting is often referred to as *multiresolution analysis* and serves as a convenient framework for the construction of *wavelet decompositions* (cf. [Mal, M]). Its intrinsic shift-invariant features allow the employment of powerful analytic tools but also suggest the whole $\mathbb{R}^s$ as the most convenient domain to work on. For most of this section, we will do so too, and formulate first the relevant norm estimates for $\Omega = \mathbb{R}^s$ in which case we continue dropping any explicit reference to the domain in our notation. The corresponding estimates for bounded domains satisfying the assumptions in Sect. 2 follow by standard extension techniques (see e.g. [JW]).

It should be mentioned that we will be dealing here with a setting similar to that discussed in [DJP] from a different point of view and for different purposes. Nevertheless, to fit several of the results in [DJP] into the present context, we will have to confirm that they still hold under somewhat less restrictive assumptions.

6.1 Refinable functions

The main ingredient of what follows will be a fixed function φ on $\mathbb{R}^s$ which will always be assumed to be at least continuous and to have compact support. In addition, φ will be required to be **a**-*refinable*. By this we mean that there exists a finitely supported *mask* $\boldsymbol{a} = \{a_\alpha : \alpha \in \mathbb{Z}^s\}$ such that

$$\varphi(x) = \sum_{\alpha \in \mathbb{Z}^s} a_\alpha \varphi(2x - \alpha), \quad x \in \mathbb{R}^s . \tag{6.1}$$

Although $\boldsymbol{a}$ will always be assumed to have *finite* support, some of the subsequent developments would remain valid for certain functions φ of unbounded support and masks $\boldsymbol{a} \in \ell_1(\mathbb{Z}^s)$, see [JM]. Moreover, φ is called *stable* if

$$\left\| \sum_{\alpha \in \mathbb{Z}^s} \lambda_\alpha \varphi(\cdot - \alpha) \right\|_p \sim \|\lambda\|_{\ell_p}, \quad \lambda \in \ell_p(\mathbb{Z}^s) . \tag{6.2}$$

We omit the reference to the particular norm ℓ_p when talking about stability because stability for some p in the range $1 \leqq p \leqq \infty$ is known to imply, under the above assumptions, stability for all $1 \leqq p \leqq \infty$ [JM].

Defining

$$V_j := \left\{ \sum_{\alpha \in \mathbb{Z}^s} \lambda_\alpha \varphi(2^j \cdot - \alpha) : \lambda \in \ell_p(\mathbb{Z}^s) \right\} , \tag{6.3}$$

it is shown in [JM] that for φ satisfying (6.1) and (6.2)

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L_p(\mathbb{R}^s), \quad 1 \leqq p \leqq \infty \tag{6.4}$$

and

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}, \quad 1 \leqq p < \infty . \tag{6.5}$$

6.2 Jackson and Bernstein Inequalities

Our first step is to establish estimates of the form (3.7) and (3.9) for the present setting.

Proposition 6.1. *Suppose $\varphi \in W^{k,1}$ has compact support and is **a**-refinable. Then, for $1 \leqq p \leqq \infty$, there exists a constant c such that*

$$\inf_{g\in V_m} \|f - g\|_p \leqq c\, 2^{-m(k+1)} |f|_{k+1,p} \tag{6.6}$$

holds for all $f \in W^{k+1,p}$.

Proof. Recall from [CDM] Theorem 8.4 that, whenever $\varphi \in W^{k,1}$, $\varphi \not\equiv 0$ satisfies (6.1), one has

$$\hat{\varphi}(0) \neq 0, \ (D^\beta \hat{\varphi})(2\pi\alpha) = 0, \quad |\beta| \leqq k, \alpha \in \mathbb{Z}^s, \ \alpha \neq 0 , \tag{6.7}$$

where

$$\hat{f}(u) = \int_{\mathbb{R}^s} f(x) \mathrm{e}^{-\mathrm{i}u\cdot x}\, dx$$

denotes the Fourier transform of f. Thus, we may assume φ to be normalized so that $\hat{\varphi}(0) = 1$ which means

$$\sum_{\alpha\in\mathbb{Z}^s} \varphi(x - \alpha) = 1, \quad x \in \mathbb{R}^s . \tag{6.8}$$

Thus we may expand

$$1/\hat{\varphi}(u) = \sum_{\beta\geqq 0} c_\beta u^\beta$$

in a neighborhood of the origin. Defining for sufficiently smooth functions f

$$(Lf)(x) := \sum_{|\beta|\leqq k} (-\mathrm{i})^{|\beta|} c_\beta (D^\beta f)(x) ,$$

it is shown in [DM2] that, whenever (6.7) holds, the operator

$$(T_m f)(x) := \sum_{\alpha\in\mathbb{Z}^s} (Lf)(2^{-m}(\cdot + \alpha))\varphi(2^m x - \alpha) \tag{6.9}$$

reproduces all polynomials of degree at most k on any bounded domain in $\mathbb{R}^s$, and satisfies for any $f \in W^{k+1,p}$

$$\|f - T_m f\|_p \leqq c\, 2^{-m(k+1)} |f|_{k+1,p} \tag{6.10}$$

whence the assertion follows. □

The key for establishing a sufficiently sharp converse estimate is the following observation.

Lemma 6.1. *Suppose* $\varphi \in C_0^k$ *is* $\boldsymbol{a}$*-refinable. Then there exists a constant c and some* $\delta > 0$, *depending only on* $\boldsymbol{a}$, *such that*

$$|(D^\beta \varphi)(x) - (D^\beta \varphi)(y)| \leqq c|x-y|_2^\delta, \quad x, y \in \mathbb{R}^s, \tag{6.11}$$

holds for all $|\beta| = k$.

Proof. We have to recall a few notions from [CDM]. Let $E = \{0, 1\}^s$ denote the set of extreme points of the unit cube $[0, 1]^s$. Suppose $\mathscr{B} = \{B_e : e \in E\}$ is a collection of $N \times N$ matrices, and let $\xi : [0, 1]^s \to \mathbb{R}^N$ be a continuous function. $(\mathscr{B}, \xi)$ is called a *refinement pair* if

$$B_e \xi(x) = \xi\left(\frac{x+e}{2}\right), \quad e \in E, \quad x \in [0, 1]^s. \tag{6.12}$$

Moreover, let $W := \operatorname{span}\{\xi(x) - \xi(y) : x, y \in [0, 1]^s\}$. Due to the equivalence of norms on finite dimensional spaces, it suffices to establish (6.11) for the sup-norm $|\cdot|_\infty$ on $\mathbb{R}^N$. By Lemma 7.3 in [CDM], there exists some $m_0 \in \mathbb{N}$ such that for $m \geqq m_0$,

$$|B_{e^1} \ldots B_{e^m} w|_\infty \leqq \tfrac{1}{2}|w|_\infty \tag{6.13}$$

for all $w \in W$ and any $e^1, \ldots, e^m \in E$. Now pick any $x, y \in [0, 1]^s$ and choose $n \in \mathbb{N}$ such that

$$2^{-n-2} \leqq |x-y|_\infty < 2^{-n-1}. \tag{6.14}$$

For

$$x = \sum_{j=1}^{\infty} 2^{-j} e^j, \qquad y = \sum_{j=1}^{\infty} 2^{-j} d^j, \quad e^j, d^j \in E, j \in \mathbb{N}$$

set

$$\tilde{x} := \sum_{j=1}^{n} 2^{-j} e^j, \qquad \tilde{y} := \sum_{j=1}^{n} 2^{-j} d^j.$$

Denoting the ith component of $\tilde{x}$ by $\tilde{x}_i$ and assuming without loss of generality that $\tilde{x}_i \geqq \tilde{y}_i$, one concludes from (6.14) that there exists some $l \in \mathbb{N}_0, l \leqq n$ such that $(e^j)_i = (d^j)_i, j = 0, \ldots, l$, $(e^{l+1})_i = 1$, $(e^j)_i = 0, j = l+2, \ldots, n$, $(d^{l+1})_i = 0$ and $(d^j)_i = 1$, $j = l+2, \ldots, n$. Hence

$$|\tilde{x} - \tilde{y}|_\infty \leqq 2^{-n}. \tag{6.15}$$

By Lemma 7.6 in [CDM], there exists $z = \sum_{j=1}^n g^j 2^{-j}, g^j \in E, j = 1, \ldots, n$, such that

$$2^n(\tilde{x} - z), \quad 2^n(\tilde{y} - z) \in [0, 1]^s. \tag{6.16}$$

Thus

$$\begin{aligned} |\xi(x) - \xi(y)|_\infty \leqq{} & |\xi(x) - \xi(\tilde{x})|_\infty + |\xi(\tilde{x}) - \xi(z)|_\infty \\ & + |\xi(z) - \xi(\tilde{y})|_\infty + |\xi(\tilde{y}) - \xi(y)|_\infty . \end{aligned} \tag{6.17}$$

A repeated application of (6.12) yields therefore

$$|\xi(x)-\xi(\tilde{x})|_\infty = |B_{e^1}\dots B_{e^n}(\xi(\hat{x})-\xi(0))|_\infty\,, \tag{6.18}$$

$$|\xi(y)-\xi(\tilde{y})|_\infty = |B_{d^1}\dots B_{d^n}(\xi(\hat{y})-\xi(0))|_\infty\,,$$

where $\hat{x}=2^n(x-\tilde{x})$, $\hat{y}=2^n(y-\tilde{y})\in[0,1]^s$.

Moreover, by (6.16) we obtain (see the proof of Lemma 7.6 in [CDM])

$$\xi(\tilde{x}) = B_{g^1}\dots B_{g^n}\xi(2^n(\tilde{x}-z))\,,$$

$$\xi(\tilde{y}) = B_{g^1}\dots B_{g^n}\xi(2^n(\tilde{y}-z)),$$

and therefore

$$|\xi(\tilde{x})-\xi(z)|_\infty = |B_{g^1}\dots B_{g^n}(\xi(2^n(\tilde{x}-z))-\xi(0))|_\infty\,, \tag{6.19}$$

$$|\xi(\tilde{y})-\xi(z)|_\infty = |B_{g^1}\dots B_{g^n}(\xi(2^n(\tilde{y}-z))-\xi(0))|_\infty\,.$$

Hence (6.17), (6.18) and (6.19) provide, in view of (6.13),

$$|\xi(x)-\xi(y)|_\infty \leqq 8 \sup_{u\in[0,1]^s} |\xi(u)|_\infty \left(\frac{1}{2}\right)^{\lfloor n/m_0\rfloor}$$

$$\leqq c\,2^{-n/m_0}\,.$$

Setting $\delta = 1/m_0$, we therefore obtain by (6.14)

$$|\xi(x)-\xi(y)|_\infty \leqq c|x-y|_\infty^\delta, \quad x,y\in[0,1]^s\,. \tag{6.20}$$

It is shown in Proposition 7.3 and Lemma 7.2 in [CDM] that the collection of matrices

$$B_e := 2^{|\eta|}(a_{e+2\nu-\beta})_{\nu,\beta\in\operatorname{supp}\boldsymbol{a}}, \quad e\in E\,,$$

together with $\xi(x) = ((D^\eta\varphi)(x+\alpha)\colon \alpha\in\operatorname{supp}\boldsymbol{a})$ forms for every $|\eta|\leqq k$ a refinement pair whenever $\varphi\in C_0^k$ is $\boldsymbol{a}$-refinable. Hence we obtain

$$|(D^\eta\varphi)(x)-(D^\eta\varphi)(y)| \leqq c|x-y|_\infty^\delta, \quad x,y\in\alpha+[0,1]^s, \quad \alpha\in\mathbb{Z}^s\,. \tag{6.21}$$

To finish the proof, note that it suffices to establish (6.1) for $x, y\in\mathbb{R}^s$, $|x-y|_\infty\leqq 1$. Whenever any two such points are located in different cubes $\alpha+[0,1]^s$, $\beta+[0,1]^s$, respectively, the line segment connecting x and y intersects the faces of the partition of $\mathbb{R}^s$ into cubes $\alpha+[0,1]^s$ M times in points z^i, $i=1,\dots,M$, say, where $M\leqq M_0$ and M_0 depends only on the spatial dimension s. Thus, setting $z^0:=x$, $z^{M+1}:=y$, (6.21) yields

$$|(D^\eta\varphi)(x)-(D^\eta\varphi)(y)| \leqq \sum_{i=1}^{M+1} |(D^\eta\varphi)(z^i)-(D^\eta\varphi)(z^{i-1})|$$

$$\leqq \sum_{i=1}^{M+1} c|z^i-z^{i-1}|_\infty^\delta \leqq c|x-y|_\infty^\delta\,.$$

This completes the proof of Lemma 6.1. □

As a consequence, we obtain

Proposition 6.2. *Suppose $\varphi \in C_0^k$ is **a**-refinable and stable* (cf. (6.1), (6.2)). *Then there exists some $\delta > 0$ such that*

$$\omega_{k+1}(g, t)_p \leqq c(\min\{1, t2^m\})^{k+\delta} \|g\|_p, \quad g \in V_m, \tag{6.22}$$

holds for some constant c independent of m.

Proof. Fix any cube $I_{m,\beta} := 2^{-m}([0,1]^s + \beta)$ and assume $t < 2^{-m}/2k$, since otherwise (6.22) follows from the trivial estimate

$$\omega_{k+1}(g, t)_p \leqq c\|g\|_p,$$

where c depends only on k. Now pick any $h \in \mathbb{R}^s$, $|h|_2 \leqq t$ and observe that

$$\begin{aligned} |\Delta_h^{k+1}\varphi(2^m x - \alpha)| &= |\Delta_h^k\varphi(2^m x + 2^m h - \alpha) - \Delta_h^k\varphi(2^m x - \alpha)| \\ &\leqq c|h|_2^k 2^{mk} \max_{|\eta| = k} |(D^\eta\varphi)(2^m y - \alpha) - (D^\eta\varphi)(2^m z - \alpha)| \end{aligned} \tag{6.23}$$

for some y, z on the line segments connecting $x + h, x + (k+1)h$ and $x, x + kh$, respectively. Here the constant c depends only on k and s. By Lemma 6.1 $D^\eta\varphi$ is Lipschitz continuous of order δ for $|\eta| = k$. Hence $|\Delta_h^{k+1}\varphi(x)| \leqq c\,|h|_2^{k+\delta}$. Since $|y - z|_2 \leqq (k+1)|h|_2$, we obtain by translation and dilation

$$|\Delta_h^{k+1}\varphi(2^m x - \alpha)| \leqq c(t2^m)^{k+\delta} \tag{6.24}$$

where the constant c depends only on s, k and δ, the constant from Lemma 6.1. Since $\operatorname{supp}(\Delta_h^{k+1}\varphi(2^m\cdot - \alpha))$ has measure $\leqq c2^{-ms}$, for some constant c depending only on k and $\boldsymbol{a}$, one obtains

$$\|\Delta_h^{k+1}\varphi(2^m\cdot - \alpha)\|_p \leqq c\,2^{-ms/p}(t2^m)^{k+\delta}. \tag{6.25}$$

Now let $g = \sum_{\alpha \in \mathbb{Z}^s} \lambda_\alpha 2^{ms/p}\varphi(2^m\cdot - \alpha) \in V_m$. Since the set $\Lambda_\beta = \{\alpha \in \mathbb{Z}^s : I_{m,\beta} \cap \operatorname{supp}(\Delta_h^{k+1}\varphi(2^m\cdot - \alpha)) \neq \emptyset\}$ has finite cardinality which can be bounded independently of $m \in \mathbb{N}$ and $\beta \in \mathbb{Z}^s$, we obtain

$$\|\Delta_h^{k+1} g\|_p(I_{m,\beta}) \leqq c \sum_{\alpha \in \Lambda_\beta} |\lambda_\alpha| 2^{ms/p} 2^{-ms/p}(t2^m)^{k+\delta},$$

and therefore

$$\|\Delta_h^{k+1} g\|_p^p(I_{m,\beta}) \leqq c\left(\sum_{\alpha \in \Lambda_\beta} |\lambda_\alpha|^p\right)(t2^m)^{(k+\delta)p}.$$

Summing over $\beta \in \mathbb{Z}^s$ yields, in view of the uniform finiteness of Λ_β,

$$\|\Delta_h^{k+1} g\|_p^p \leqq c(t2^m)^{(k+\delta)p}\left(\sum_{\alpha \in \mathbb{Z}^s} |\lambda_\alpha|^p\right). \tag{6.26}$$

Since the stability estimate (6.2) is equivalent to the fact that the relation

$$\|\lambda\|_{\ell_p} \sim \Big\|\sum_{\alpha \in \mathbb{Z}^s} \lambda_\alpha 2^{ms/p}\varphi(2^m\cdot - \alpha)\Big\|_p \tag{6.27}$$

holds uniformly in $m \in \mathbb{N}_0$, the claim (6.22) follows now, in view of the definition of g, from (6.26) and (6.27). □

Note that the smoothness condition on φ in Proposition 6.2 is slightly stronger than that in Proposition 6.1. The advantage of requiring $\varphi \in C^k$ is that Lemma 6.1 automatically guarantees some extra Hölder continuity. Moreover, relatively simple conditions on the mask $\boldsymbol{a}$, for instance in terms of factorizations of the symbol of $\boldsymbol{a}$, are available that guarantee the derivatives of φ up to a certain degree to be continuous [CDM].

6.3 Discrete norms and Besov spaces

We adhere to the notation in Sect. 4 and assume again that $\mathscr{Q}$ is a sequence of linear projectors Q_j mapping any V_m, $m \geqq j$, onto V_j. The corresponding discrete norms are then defined in a completely analogous fashion with $W_j := (Q_j - Q_{j-1})V_j$. In view of Proposition 6.1 and Proposition 6.2, Theorem 4.1 and Proposition 4.1 yield

Corollary 6.1. *Let $\varphi \in C_0^k$ be $\boldsymbol{a}$-refinable and stable. Then, for each $0 < r < k + \delta$ there exist constants $0 < c_1, c_2 < \infty$ independent of $g \in V_m$, $m \in \mathbb{N}_0$, such that*

$$(6.28) \qquad \frac{c_1}{v_m^{(p)}} \|g\|_{(r,m,p,q)} \leqq \|g\|_{B^r_{p,q}} \leqq c_2 \|g\|_{(r,m,p,q)}, \quad g \in V_m ,$$

where $\delta > 0$ depends only on $\boldsymbol{a}$.

Corollary 6.2. *Suppose that, in addition to the assumptions in Corollary 6.1, the Q_j are uniformly bounded on L_p. Then*

$$\|\cdot\|_{(r,\infty,p,q)} \sim \|\cdot\|_{B^r_{p,q}} \quad \textit{for } 0 < r < k + \delta .$$

It is perhaps worthwhile relating the above observations to a concept discussed for instance in [O1] in connection with finite element spaces. To this end, we introduce the spaces $A^r_{p,q} = A^r_{p,q}(\varphi)$, $r > 0$, $1 \leqq p, q \leqq \infty$, consisting of all elements f in L_p such that

$$\|f\|_{A^r_{p,q}} := \inf\{ \|\{2^{jr}\|g_j\|_p\}_{j\in\mathbb{N}_0}\|_{\ell_q} : f = \sum_{j=0}^{\infty} g_j,\ g_j \in V_j\} < \infty .$$

Clearly, $A^r_{p,q}(\varphi)$ equipped with this norm is a Banach space.

Defining again

$$E_m(f)_p := \inf_{g \in V_m} \|f - g\|_p ,$$

we note next

Remark 6.1. The expression

$$\|f\|_{p,q,r} := \|f\|_p + \|\{2^{rm} E_m(f)_p\}_{m\in\mathbb{N}_0}\|_{\ell_q}$$

defines an equivalent norm on $A^r_{p,q}$.

In fact, the estimate

$$(6.29) \qquad \|f\|_{A^r_{p,q}} \leqq c \|f\|_{p,q,r}$$

follows readily by choosing $g_j = P_j - P_{j-1}$, $P_{-1} = 0$, where P_j is a best approximation to f from V_j. Conversely, observe that for $\mu \leqq 1$

$$E_m(f)_p \leqq \left\| \sum_{j=m+1}^{\infty} g_j \right\|_p \leqq \left(\sum_{j=m+1}^{\infty} \|g_j\|_p^\mu \right)^{1/\mu}$$

is valid for any representation $f = \sum_{j=0}^{\infty} g_j$. The latter estimate combined with a discrete Hardy type inequality yields the converse inequality to (6.29), thereby proving the above claim. □

Recall next that, under certain circumstances, $\|\cdot\|_{p,q,r}$ establishes a norm for the Besov space $B^r_{p,q}$ [DP1]. The equivalence of the norms $\|\cdot\|_{p,q,r}$ and $\|\cdot\|_{B^r_{p,q}}$ has already been established in [DJP] for a similar setting, however under slightly more restrictive assumptions on φ and for a somewhat smaller range of r.

Corollary 6.3. *Suppose $\varphi \in C_0^k$ is **a**-refinable and stable. Then there exists some $0 < \delta \leqq 1$, depending only on **a** such that*

$$B^r_{p,q} = A^r_{p,q}(\varphi), \quad 0 < r < k + \delta, \quad 1 \leqq p, q \leqq \infty ,$$

and $\|\cdot\|_{B^r_{p,q}} \sim \|\cdot\|_{A^r_{p,q}}$.

Proof. The Jackson estimate (6.6) implies, as in the proof of Propositition 4.1, the estimate (4.15) which immediately yields

$$B^r_{p,q} \subseteq A^r_{p,q}(\varphi)$$

for the required range of r, p, q with a corresponding norm estimate. In view of the Bernstein inequality in Proposition 6.2, the converse estimate is an immediate consequence of the estimates following (4.7) in the proof of Theorem 4.1 with $\gamma = k + \delta$. □

6.4 Wavelets

In view of Theorem 3.1, uniformly bounded condition numbers are obtained when $v_m = O(1)$, $m \to \infty$. By (6.6) and Proposition 4.1 this is, for instance, the case when $\mathcal{Q}$ is uniformly bounded on L_2. Thus, a natural choice for Q_j would be the orthogonal projection of L_2 onto V_j so that

(6.30) $$V_{j-1} \perp W_j, \quad j \in \mathbb{Z} ,$$

and

(6.31) $$\|Q_j f - f\|_2 = E_j(f)_2 .$$

This choice corresponds to the Bramble-Pasciak-Xu preconditioner. Now an important issue is to find appropriate bases for the spaces W_j. One expects the W_j's to be spanned by the integer translates of $2^s - 1$ functions of the form

(6.32) $$\psi_e(x) = \sum_{\alpha \in \mathbb{Z}^s} a^e_\alpha \varphi(2x - \alpha), \quad e \in E^* := \{0,1\}^s \backslash \{0\} = E \backslash \{0\} .$$

In [JM] (see e.g. Theorem 6.1 there) necessary and sufficient conditions (in terms of the mask $\boldsymbol{a}$) are developed when for a given compactly supported $\boldsymbol{a}$-refinable stable φ, there exist additional finitely supported masks $\boldsymbol{a}^e$, $e \in E^*$, so that the functions

$$\psi_{j,e,\alpha}(x) := 2^{js/2}\psi_e(2^j x - \alpha), \quad e \in E^*, \ \alpha \in \mathbb{Z}^s, \tag{6.33}$$

(being also compactly supported) form a stable basis for W_{j+1} in the sense of (3.10) with $c_j = 1$. Moreover, for $s = 1, 2, 3$ explicit expressions for the masks $\boldsymbol{a}^e$, $e \in E^*$, in terms of the mask $\boldsymbol{a}$ of φ are derived in [RS] under the assumption that φ is *skew-symmetric* which means that for some $y \in \mathbb{R}^s$,

$$\varphi(y + x) = \overline{\varphi(y - x)}, \quad x \in \mathbb{R}^s, \tag{6.34}$$

(see also [CSW]). In fact, the refinement equation (6.1) forces y to belong to $\frac{1}{2}\mathbb{Z}^s$. Examples of functions φ satisfying (6.34) are box splines (see [BH, DM1] for more details) which, in turn, cover tensor product B-splines as special cases.

The functions $\psi_{j,e,\alpha}$ with the above properties are usually referred to as *pre-wavelets*. Of course, they include the case that even the translates $\psi(\cdot - \alpha)$ are orthonormal within a given level so that

$$\int_{\mathbb{R}^s} \psi_{j,e,\alpha}(x)\psi_{k,e',\beta}(x)dx = \delta_{j,k}\delta_{e,e'}\delta_{\alpha,\beta}. \tag{6.35}$$

The *wavelets* $\psi_{j,e,\alpha}$ are then trivially stable and form a complete orthonormal system for L_2.

While there are several realizations of pre-wavelets, the construction of (orthogonal) multivariate wavelets appears to be much harder. However, tensor products of the compactly supported univariate orthonormal wavelets constructed in [Dau] do, of course, provide orthonormal multivariate wavelets with compact support and any degree of regularity.

On the other hand, choosing the Q_j to be orthogonal projections is by no means the only way to guarantee uniformly bounded v_m. An alternative, and presumably more flexible concept is offered by *biorthogonal* wavelets. For a detailed analysis of such univariate constructions we refer to [CDF]. A general multivariate setting may be formulated as follows. Suppose that in addition to φ satisfying (6.1), there exists a function $\zeta \in L_2(\mathbb{R}^s)$ of compact support which satisfies

$$\zeta(x) = \sum_{\alpha \in \mathbb{Z}^s} d_\alpha \zeta(2x - \alpha), \tag{6.36}$$

where the mask $\boldsymbol{d}$ is again finitely supported, such that

$$\langle \varphi, \zeta(\cdot - \alpha) \rangle := \int_{\mathbb{R}^s} \varphi(x)\overline{\zeta(x - \alpha)}dx = \delta_{0,\alpha}, \quad \alpha \in \mathbb{Z}^s. \tag{6.37}$$

Clearly, $\zeta = \varphi$, $\boldsymbol{d} = \boldsymbol{a}$ recovers again the case of orthogonal wavelets. The objective is then to find additional masks $\boldsymbol{a}^e, \boldsymbol{d}^e$, $e \in E^*$, such that the functions

$$\begin{aligned} \psi_e(x) &:= \sum_{\alpha \in \mathbb{Z}^s} a^e_\alpha \varphi(2x - \alpha) \\ \zeta_e(x) &:= \sum_{\alpha \in \mathbb{Z}^s} d^e_\alpha \zeta(2x - \alpha) \end{aligned} \tag{6.38}$$

satisfy the *biorthogonality conditions*

(6.39) $$\langle \psi_e, \zeta_{e'}(\cdot - \alpha)\rangle = \delta_{e,e'}\delta_{0,\alpha}, \quad e, e' \in E, \alpha \in \mathbb{Z}^s,$$

where $E = E^* \cup \{0\}$ and $\psi_0 := \varphi, \zeta_0 := \zeta$. (For the construction of $\boldsymbol{d}^e, \boldsymbol{a}^e, e \in E^*$, see [DM4].) It is not hard to prove that the mappings

$$(Q_j f)(x) := \sum_{\alpha \in \mathbb{Z}^s} \langle f, \zeta(2^j \cdot - \alpha)\rangle 2^{js}\varphi(2^j x - \alpha)$$

are *uniformly bounded* projectors from L_2 onto V_j, and that the complements

$$W_{j+1} := (Q_{j+1} - Q_j)V_{j+1}$$

are spanned by the functions

(6.40) $$\psi_{j,e,\alpha}(x) = 2^{js/2}\psi_e(2^j x - \alpha), \quad e \in E^*, \alpha \in \mathbb{Z}^s,$$

which form stable bases (with $c_j = 1$) for the spaces W_{j+1}.

To summarize these findings, let us suppose throughout the remainder of this section that $\varphi \in C_0^k$ is $\boldsymbol{a}$-refinable, stable and admits the construction of (pre-)wavelets or biorthogonal wavelets $\psi_{j,e,\alpha}$ in the above sense. Then every $f \in L_2$ has a unique representation

(6.41) $$f(x) = \sum_{\alpha \in \mathbb{Z}^s} \lambda_\alpha(f)\varphi(x - \alpha) + \sum_{j=0}^{\infty} \sum_{e \in E^*} \sum_{\alpha \in \mathbb{Z}^s} \lambda_{j,e,\alpha}(f)\psi_{j,e,\alpha}(x),$$

and, setting $\lambda(f) = \{\lambda_\alpha(f) \colon \alpha \in \mathbb{Z}^s\}$, $\lambda^{(j)}(f) = \{\lambda_{j,e,\alpha}(f) \colon e \in E^*, \alpha \in \mathbb{Z}^s\}, j \in \mathbb{N}_0$, one has

(6.42) $$\left(\|\lambda(f)\|_{\ell_2}^2 + \sum_{j=0}^{\infty} \|\lambda^{(j)}(f)\|_{\ell_2}^2\right) \sim \|f\|_2^2.$$

Moreover, Corollary 6.2 and (4.10) yield the following characterization of Sobolev spaces.

Corollary 6.4. *For φ as above, one has*

(6.43) $$\left(\|\lambda(f)\|_{\ell_2}^2 + \sum_{j=0}^{\infty} 2^{2kj}\|\lambda^{(j)}(f)\|_{\ell_2}^2\right) \sim \|f\|_{k,2}^2.$$

6.5 Bounded domains

To draw any conclusions from the estimates in Corollaries 6.1, 6.2 or 6.4 on growth rates of condition numbers, one has to consider the case of bounded domains. We wish to briefly address several instances where the above estimates apply essentially as they are.

I. Periodic boundary conditions. Suppose $\Omega = [0, 1]^s$ (or any hyperrectangle), and one looks for 1-periodic solutions of (2.1), i.e., for functions u satisfying

$$u(x) = u(x + \alpha), \quad \alpha \in \mathbb{Z}^s.$$

Following [DPS], we identify such functions with functions on the s-dimensional torus

$$\prod^s := \mathbb{R}^s/\mathbb{Z}^s ,$$

so that the periodization operator

$$[f](x) := \sum_{\alpha\in\mathbb{Z}^s} f(x+\alpha)$$

maps $L_2(\mathbb{R}^s)$ into $L_2(\prod^s)$. Likewise, one may identify for notational convenience the cosets $[x] := x + \mathbb{Z}^s$, $x\in\mathbb{R}^s$, with its representer in $[0,1]^s$. For a given refinable function φ and corresponding pre-wavelets or biorthogonal wavelets ψ_e, $e\in E^*$, let

$$\tilde{\psi}_{j,0,\alpha} = \tilde{\varphi}_{j,\alpha} := 2^{js/2}[\varphi(2^j\cdot-\alpha)],\quad \tilde{\psi}_{j,e,\alpha} := [\psi_{j,e,\alpha}],\quad j\in\mathbb{N}_0, e\in E^*, \alpha\in\mathbb{Z}^s ,$$

where the $\psi_{j,e,\alpha}$ are given by (6.40), and define

$$S_j := \operatorname{span}\{\tilde{\varphi}_{j,\alpha} : \alpha\in\mathbb{Z}^{s,j}\} ,$$

where

$$\mathbb{Z}^{s,j} := \mathbb{Z}^s/(2^j\mathbb{Z}^s) .$$

As pointed out in [DPS], the spaces S_j are nested and form a multiresolution analysis on $L_2(\prod^s)$, provided φ generates a multiresolution analysis on $L_2(\mathbb{R}^s)$. Moreover, the original orthogonality conditions like (6.30), (6.35), or (6.39) carry over to analogous conditions relative to the inner product $\langle f, g\rangle_\Pi := \int_{[0,1]^s} f(x)\overline{g(x)}\,dx$, as long as one shifts in $\mathbb{Z}^{s,j}$.

It is clear that the corresponding Jackson and Bernstein estimates remain valid, and so do the above norm estimates relative to the corresponding 1-periodic Besov spaces. Hence we infer from Theorem 3.2 or from (6.40) and Corollary 3.1 combined with Corollary 6.4, that, for φ as above, the stiffness matrices M_m relative to the basis

$$\{\tilde{\varphi}_{0,\alpha} : \alpha\in\mathbb{Z}^{s,0}\} \cup \{\tilde{\psi}_{j,e,\alpha} : e\in E^*, \alpha\in\mathbb{Z}^{s,j}, j=0,\dots,m\}$$

satisfy

$$\kappa(D^{-1}M_m D^{-1}) = O(1),\quad m\to\infty , \tag{6.44}$$

where the diagonal matrix D is defined by $D_{(j,e,\alpha),(j',e',\alpha')} = 2^{kj}\delta_{j,j'}\delta_{e,e'}\delta_{\alpha,\alpha'}$.

II. Homogeneous boundary conditions. When in (2.1) $Bu = (D^\alpha u : |\alpha| \leqq k) = 0$ on $\partial\Omega$ (in the sense of traces), one could define $S_m \subset V_m$ to be the span of all translates $\varphi(2^m\cdot-\alpha)$ whose support is contained in Ω so that $S_m \subset H_0^k(\Omega)$. Again uniform boundedness (6.44) is obtained. However, if the supports of the $\varphi(2^m\cdot-\alpha)$ do not match the boundary in a proper way, one expects that the accuracy of such a scheme deteriorates near the boundary.

It is therefore still important to consider

III. The general case. A possible choice would be to define $S_m \subset V_m$ to be spanned by those translates $\varphi(2^m\cdot-\alpha)$ whose support intersects Ω. Under our general assumption on Ω, the characterization of the corresponding local Besov spaces remains the same (cf. [DP1]). Using extension arguments, the Jackson and Bernstein estimates remain valid as well. Thus the above estimates still apply provided that the particular way of enforcing (essential) boundary conditions is compatible with the ellipticity of the problem. These issues will be studied in more detail in a forthcoming paper.

6.6 Computational aspects

We will conclude this section with briefly commenting on some principle computational consequences arising from the refinability of the generating function φ.

The first issue is to compute the entries of the stiffness matrices A_m relative to the basis functions $\varphi(2^m \cdot - \alpha)$ on the finest level (which corresponds to a 'nodal' basis), as well as of the stiffness matrices M_m relative to the multilevel basis. Concerning A_m (ignoring again for the moment boundary effects), this amounts to computing quantities of the form

$$\int_{\mathbb{R}^s} c_{\eta,\beta}(x)(D^\eta\varphi)(2^m x - \alpha)(D^\beta\varphi)(2^m x - \alpha')\,dx\,. \tag{6.45}$$

It is shown in [DM3] that the computation of even more general expressions of the form

$$\int_{\mathbb{R}^s} \prod_{i=1}^{n} (D^{\eta^i}\varphi_i)(2^{m_i}x - \alpha^i)\,dx\,, \tag{6.46}$$

where each φ_i is $\boldsymbol{a}_i$-refinable, say, and the m_i are possibly different integers, reduces, for each fixed collection of φ_i's and η^i's, to the solution of a single *eigenvector-moment* problem, independent of the m_i, α^i, whose size depends only on the support of the φ_i's. Of course, when the coefficients $c_{\eta,\beta}(x)$ are actually constant, this method applies for $n = 2$, $m_1 = m_2$ and $\varphi_1 = \varphi_2$ in (6.46). For non-constant coefficients, one could choose a possibly different refinable function ϕ along with some local approximation scheme

$$(B_{m'}f)(x) := \sum_{\alpha\in\mathbb{Z}^s} F_{m',\alpha}(f)\phi(2^{m'}x - \alpha)\,,$$

where the $F_{m',\alpha}$ are suitable functionals supported in a small neighborhood of $2^{-m'}\alpha$, and replace $c_{\eta,\beta}(x)$ in (6.45) by $(B_{m'}c_{\eta,\beta})(x)$. Such schemes are well understood when ϕ is for instance a tensor product B-spline or a box spline (see [BH, DM1, DM2]). Here m' has to be chosen so that the overall accuracy is not decreased by the resulting 'quadrature error'. In the case of variable coefficients, one has to compute then quantities of type (6.46) with $n = 3$, $m_1 = m_2 = m$, $m_3 = m'$ and $\varphi_1 = \varphi_2 = \varphi, \varphi_3 = \phi, \eta_3 = 0$. The same strategy or an iterative procedure also pointed out in [DM3] could be used to compute the entries of the right hand side of the linear system. At any rate, the refinability of φ allows to solve these tasks very efficiently so that the matrix A_m would be available at relatively low cost.

Using (6.32), the entries of M_m could be computed in the same fashion. On the other hand, it is not difficult to determine the matrix L_m in (2.10) which performs the corresponding change of bases and yields the preconditioner $C_m = L_m L_m^{\mathrm{T}}$. To describe this, let Ω_m denote the grid of level m consisting of points of the form $2^{-m}\alpha$, $\alpha\in\mathscr{G}_m\subset\mathbb{Z}^s$. Ω_m typically contains all points in $2^{-m}\mathbb{Z}^s$ whose distance from Ω is of the order at most 2^{-m}. This set agrees, except perhaps near the boundary with

$$\{(2\beta+e)/2^m : e\in E, \beta\in\mathscr{G}_{m-1}\}\ .$$

Denoting for the sake of convenience $\boldsymbol{a}^0 := \boldsymbol{a}$ and adopting this ordering for the columns in our matrices, we define first a matrix $L_m^{(1)}$ by setting

$$(L_m^{(1)})_{(\alpha,2\beta+e)} := a^e_{\alpha-2\beta},\quad \beta\in\mathscr{G}_{m-1}, e\in E, \alpha\in\mathscr{G}_m\ . \tag{6.47}$$

Now the relations (6.1) and (6.32) just mean that

$$(L_m^{(1)})^{\mathrm{T}} A_m L_m^{(1)}$$

is the stiffness matrix relative to the basis consisting of $\varphi(2^{m-1}\cdot-\beta)$, $\psi_{m-1,e,\beta}$, $e\in E^*$, $\beta\in\mathscr{G}_{m-1}$, i.e., of the (nodal) basis for S_{m-1} and the basis of W_m. Analogously, we form the matrix $\tilde{L}_m^{(2)}$ relative to $\mathscr{G}_{m-1}$ and then extend $\tilde{L}_m^{(2)}$ to a matrix of size $\#\mathscr{G}_m$ by setting

$$(L_m^{(2)})_{2\alpha,2\alpha'} := (\tilde{L}_m^{(2)})_{\alpha,\alpha'},\quad \alpha,\alpha'\in\mathscr{G}_{m-1}\ ,$$

while

$$(L_m^{(2)})_{2\alpha+e,2\alpha'+e'} = \delta_{\alpha,\alpha'}\delta_{e,e'},\quad \alpha,\alpha'\in\mathscr{G}_{m-1}, e,e'\in E^*\ .$$

Forming the matrices $L_m^{(j)}$, $j=3,4,\ldots,m$, in a completely analogous fashion, we obtain

$$L_m = L_m^{(1)}\ldots L_m^{(m)}$$

as the desired transformation matrix.

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