BPX-TYPE PRECONDITIONERS FOR 2ND AND 4TH ORDER ELLIPTIC PROBLEMS ON THE SPHERE*

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Abstract. We develop two Bramble–Pasciak–Xu-type preconditioners for second resp. fourth order elliptic problems on the surface of the two-sphere. To discretize the second order problem we use C^0 linear elements on the sphere, and for the fourth order problem we use C^1 finite elements of Powell–Sabin type on the sphere. The main idea why these BPX preconditioners work depends on this particular choice of basis. We prove optimality and provide numerical examples. Furthermore we numerically compare the BPX preconditioners with the suboptimal hierarchical basis preconditioners.

Key words. BPX preconditioner; C^0 and C^1 finite elements; elliptic equations on surfaces.

AMS subject classifications. 65F10, 65F35, 65N30, 35J20, 35J35

1. Introduction. The aim of the present paper is the development of two Bramble–Pasciak–Xu (BPX) [6] preconditioners for second resp. fourth order elliptic problems on the two-dimensional sphere. Such problems arise from several applications in physical geodesy, oceanography and meteorology, [7]. The geometry of the sphere is a major obstacle in constructing suitable approximation spaces for solving partial differential equations. Often a transformation into spherical coordinates is used which gives rise to singularities at the "poles" of the sphere. Therefore, an important point in our method is the use of homogeneous polynomials in $\mathbb{R}^3$. In order to develop the theory we shall restrict ourselves to the following two most simple equations

$$-\Delta_S u = f \quad \text{on} \quad S, \tag{1.1}$$

and

$$\Delta_S^2 u = f \quad \text{on} \quad S, \tag{1.2}$$

where Δ_S is the Laplace–Beltrami operator on the two-sphere S. In order to work with Cartesian coordinates we write down the Laplace–Beltrami operator in terms of the tangential gradient

$$\nabla_S u := \nabla u - (n \cdot \nabla u) n,$$

with n the outward normal to S. The Laplace–Beltrami operator on S can now be defined as

$$\Delta_S := \nabla_S \cdot \nabla_S.$$

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We use C^0 continuous piecewise linear spherical polynomials to discretize the variational problem

$$\int_S \nabla_S u \nabla_S v \; d\omega = \int_S fv \; d\omega \quad \text{for all } v \in H^1(S) \tag{1.3}$$

corresponding to (1.1), and C^1 continuous piecewise quadratic spherical polynomials to discretize the variational problem

$$\int_S \Delta_S u \Delta_S v \; d\omega = \int_S fv \; d\omega \quad \text{for all } v \in H^2(S) \tag{1.4}$$

corresponding to (1.2). For every $f \in L_2(S)$ with $\int_S f \; d\omega = 0$ there exists a weak solution $u \in H^1(S)$ of (1.3) and a weak solution $u \in H^2(S)$ of (1.4). In both cases u is unique up to a constant, see e.g. [4, 14].

So let $k \in \{1, 2\}$, and suppose $V \subset H^k(S)$ is a space of conforming C^{k-1} finite elements defined on a spherical triangulation of S with mesh size h. Define $a(u, v)$ as the bilinear form induced by (1.3) resp. (1.4) given $k = 1$ resp. $k = 2$, and let A denote the positive definite selfadjoint operator on V defined by

$$a(u, v) = (Au, v), \quad v \in H^k(S), \tag{1.5}$$

where $(\cdot, \cdot)$ denotes the inner product of $L_2(S)$. Then we have to solve the linear operator equation

$$Au = b \tag{1.6}$$

for some $u \in V$, where $b \in V$ is defined by $(b, v) = (f, v)$, $v \in V$. The conjugate gradient method is a very efficient solver for large linear systems arising from problems such as (1.6). However, because of stability reasons, it is necessary that these systems have been suitably preconditioned. It is a known fact (see, e.g., [10]) that if for some constants $0 < \gamma, \Gamma < \infty$ and some invertible operator C

$$\gamma(C^{-1}u, u) \le a(u, u) \le \Gamma(C^{-1}u, u), \quad u \in V, \tag{1.7}$$

then the spectral condition number $\kappa(C^{1/2}AC^{1/2})$ is bounded by Γ/γ.

Let us represent the operator A by the stiffness matrix $A_\Phi := (a(\phi_i, \phi_j))_{i,j\in I}$ with respect to some typical nodal basis $\Phi := \{\phi_i \; : \; i \in I\}$ of V. Then it is known that $\kappa(A_\Phi) = \mathcal{O}(h^{-2})$ for the problem (1.3) and $\kappa(A_\Phi) = \mathcal{O}(h^{-4})$ for the problem (1.4). In order to precondition the system

$$A_\Phi y = b_\Phi, \quad (b_\Phi)_i := (f, \phi_i), \; i \in I,$$

one can perform a change of basis. So let $\Psi = \{\psi_i \; : \; i \in I\}$ be another basis of V, and L be the transfer matrix between the two bases. Then

$$A_\Psi = L^T A_\Phi L,$$

which suggests the use of $C = LL^T$ as preconditioner for the nodal basis discretization.

Several approaches exist to construct a suitable preconditioner, such as the hierarchical basis preconditioner [26] and the closely related BPX preconditioner [6]. The growth rate of the condition numbers was shown to be logarithmic in the size of the problem for the hierarchical basis preconditioner ([26]), and uniformly bounded for the BPX preconditioner in [10, 22]. Originally, these results were formulated for second order problems on two-dimensional planar domains, but they could also be established for fourth order problems on the plane, [12, 16, 21]. Recently, we constructed a hierarchical basis preconditioner for fourth order elliptic problems on the surface of the sphere in [18]. The growth rate of the condition number was shown to be logarithmic which is, as expected, similar to the planar case. It is the aim of the present paper to prove optimality of a BPX preconditioner for the problems (1.3) and (1.4), independent of the discretization, and to give numerical evidence of this optimality. We emphasize that the crucial steps in the optimality proof depend on the particular choice of basis, and, thus, are not valid for arbitrary C^0 or C^1 finite elements on the sphere. For both problems we explicitly construct a suitable basis that is easy to implement.

The outline of the remaining sections is as follows. In Section 2, we introduce the C^0 continuous piecewise linear and C^1 continuous piecewise quadratic spherical polynomials that will be used to discretize the problem (1.3) resp. (1.4). The corresponding BPX preconditioners are constructed in Section 3 and we prove their optimality. Finally, in Section 4 we conclude with some numerical experiments that confirm the theory with small absolute condition and iteration numbers.

We finish this introduction with a note about notation. We always mean by $a \sim b$ that $a \lesssim b$ and $a \gtrsim b$ hold, where $a \lesssim b$ means that a can be bounded by a constant multiple of b uniformly in any parameters on which a,b may depend, and $a \gtrsim b$ means $b \lesssim a$.

2. Suitable elements on the sphere. In a series of papers [1, 2, 3], Alfeld *et al.* develop spline spaces on triangulations on the sphere analogous to the classical spline spaces on planar triangulations. The idea is to work with homogeneous Bernstein-Bézier polynomials in $\mathbb{R}^3$ which are then restricted to the sphere. A function f defined on $\mathbb{R}^3$ is homogeneous of degree d provided that $f(\alpha v) = \alpha^d f(v)$ for all real α and all $v \in \mathbb{R}^3$. The space $\mathbb{H}_d$ of trivariate polynomials of degree d that are homogeneous of degree d is a $\binom{d+2}{2}$ dimensional subspace of the space of trivariate polymials of degree d. If we restrict $\mathbb{H}_d$ to any hyperplane H in $\mathbb{R}^3$, then we just recover the well-known space of bivariate polynomials on H. Similarly, let Ω be any subset of the sphere S, then we define $\mathbb{H}_d(\Omega)$ as the restriction of $\mathbb{H}_d$ to Ω and we refer to it as the *space of spherical polynomials* of degree d on Ω. Let Δ be a conforming spherical triangulation of $\Omega \subseteq S$, cfr. [24]. Then we define the space of spherical splines of degree d and smoothness r associated with Δ as

$$S_d^r(\Delta) := \{s \in C^r(S) \mid s|_\tau \in \mathbb{H}_d(\tau),\ \tau \in \Delta\},$$

where $s|_\tau$ denotes the restriction of s to the spherical triangle τ.

Troughout the paper we assume that some initial triangulation Δ_0 of S is given and that

$$\Delta_0 \subset \Delta_1 \subset \cdots \subset \Delta_l \subset \cdots, \quad l = 0, 1, \ldots, \tag{2.1}$$

is a sequence of ρ-adic refined triangulations obtained by subdividing the triangles at level l (i.e. the triangles of Δ_l) into ρ^2 subtriangles of level $l+1$ such that the minimum angle condition is satisfied, and such that

$$\operatorname{diam} \tau \sim \rho^{-l}, \quad \tau \in \Delta_l, \quad l = 0, 1, \ldots.$$

For each $l = 0, 1, \ldots$ we define $P_{i,l}$, $i = 1, \ldots, N_l$, as the vertices of the triangulation Δ_l.

2.1. C^0 linear elements on the sphere. The C^0 continuous piecewise linear spherical polynomials that we describe here are a natural extension of the well-known linear elements introduced by Courant [9]. However, our approach differs significantly from previous constructions (e.g., [5, 14]). We create suitable basis functions for the nested spherical spline spaces

$$S_1^0(\Delta_0) \subset S_1^0(\Delta_1) \subset \cdots \subset S_1^0(\Delta_l) \subset \cdots, \quad l = 0, 1, \ldots,$$

and this approach allows us to point out a strong connection with the classical Courant elements on the plane.

So let us define a nodal basis for $S_1^0(\Delta_l)$ by solving the following interpolation problem: find functions $\phi_{i,l} \in S_1^0(\Delta_l)$, $i = 1, \ldots, N_l$, such that $\phi_{i,l}(P_{k,l}) = \delta_{i,k}$. In fact, we can look at each spherical basis functions $\phi_{i,l}$ as the restriction of a trivariate homogeneous function to the sphere S. In particular, let f be any spherical function and $d \in \mathbb{N}$, then we define $(f)_d$ as its homogeneous extension of degree d, i.e.

$$(f)_d(v) := |v|^d f\left(\frac{v}{|v|}\right), \quad v \in \mathbb{R}^3 \setminus \{0\}. \tag{2.2}$$

Clearly we have that $\phi_{i,l} \equiv (\phi_{i,l})_1|_S$. Moreover, if we restrict $(\phi_{i,l})_1$ to the tangent plane touching S at $P_{i,l}$, then we just get a classical Courant element defined on this tangent plane centered around the vertex $P_{i,l}$. This idea can be exploited to extend several properties of the classical Courant elements to the spherical elements $\phi_{i,l}$. The following lemma is obvious.

LEMMA 2.1. *The nodal basis functions* $\{\phi_{i,l} \mid i = 1, \ldots, N_l\}$ *satisfy*

$$\|\sum_{i=1}^{N_l} c_{i,l}\phi_{i,l}\|_{L_\infty} \sim \max_i |c_{i,l}|.$$

Proof. There exists a triangle $\tau \in \Delta_l$ and a point $v \in \tau$ such that

$$\|\sum_{i=1}^{N_l} c_{i,l}\phi_{i,l}\|_{L_\infty} = \left|\sum_{i=1}^{N_l} c_{i,l}\phi_{i,l}(v)\right| \le \max_i |c_{i,l}| \sum_{i|P_{i,l}\in\tau} \|\phi_{i,l}\|_{L_\infty} \lesssim \max_i |c_{i,l}|.$$

The other inequality follows from $|c_{k,l}| = \left|\sum_{i=1}^{N_l} c_{i,l}\phi_{i,l}(P_{k,l})\right| \le \|\sum_{i=1}^{N_l} c_{i,l}\phi_{i,l}\|_{L_\infty}$. ∎

To derive the optimality of the BPX preconditioner we will need the following theorem.

THEOREM 2.2. *For any* $1 < p < \infty$ *we have*

$$\|\sum_{i=1}^{N_l} c_{i,l}\phi_{i,l}\|_{L_p}^p \sim \rho^{-2l} \sum_{i=1}^{N_l} |c_{i,l}|^p.$$

Proof. Since we have already established Lemma 2.1, the proof is identical to the corresponding proof for the classical Courant elements on the plane from [9]. ∎

2.2. C^1 Powell–Sabin elements on the sphere. In general, maintaining C^1 continuity conditions between neighbouring triangles results in non-trivial relations, see, e.g., [15]. Therefore, to overcome this problem, we will focus on the Powell–Sabin

6-split of a triangulation. Starting from an arbitrary spherical triangulation Δ, we introduce further substructures by subdividing each triangle of Δ into 6 subtriangles as shown in Figure 2.1. We will refer to this new triangulation as Δ^{PS}. The spline space $S_2^1(\Delta^{PS})$ of piecewise quadratic C^1 spherical polynomials over Δ^{PS} will be called the space of *spherical Powell–Sabin splines*. This spline space has dimension $3N$ with N the number of vertices in Δ. A spline function $s \in S_2^1(\Delta^{PS})$ will be uniquely defined by its function values and tangential gradients at the vertices of the triangulation Δ. We refer to [18] for all the details.

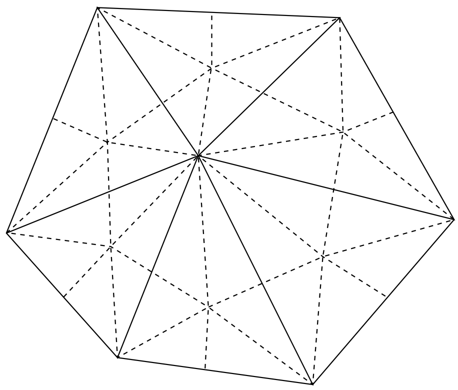

FIG. 2.1. *A PS 6-split of a triangulation* Δ.

We assume that the corresponding Powell–Sabin 6-splits of (2.1) are also nested, i.e.

$$\Delta_0^{PS} \subset \Delta_1^{PS} \subset \cdots \subset \Delta_l^{PS} \subset \cdots, \quad l = 0, 1, \ldots, \tag{2.3}$$

and that

$$\operatorname{diam} \tau \sim \rho^{-l}, \quad \tau \in \Delta_l^{PS}, \quad l = 0, 1, \ldots.$$

In this way we get a sequence of Powell–Sabin spline spaces $S_2^1(\Delta_l^{PS})$ that are nested. In most applications one desires a dyadic refinement procedure, i.e. $\rho = 2$, but in order to satisfy (2.3) severe regularity restrictions have to be imposed on the initial triangulation Δ_0. Another possibility is triadic subdivision, i.e. $\rho = 3$. Here (2.3) can be satisfied for arbitrary initial triangulations Δ_0, see [25]. Figure 2.2 demonstrates the principle.

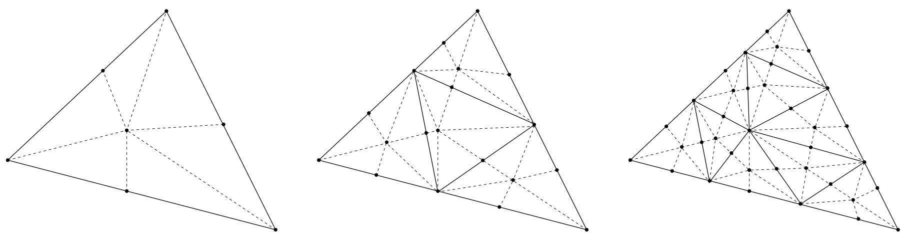

FIG. 2.2. *Dyadic and triadic subdivision.*

With each vertex $P_{i,l} \in \Delta_l$ we associate two directions $g_{i,l}$ and $h_{i,l}$ such that the set $(P_{i,l}, g_{i,l}, h_{i,l})$ forms an orthonormal basis for $\mathbb{R}^3$. For instance, suppose that $P_{i,l}$

has spherical coordinates $(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$, $\theta \in [0, 2\pi]$, $\phi \in [0, \pi]$, then take $g_{i,l} = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)$ and $h_{i,l} = (-\sin\theta, \cos\theta, 0)$. Let us introduce the functionals

$$\lambda^1_{i,l}(f) := f(P_{i,l}), \quad \lambda^2_{i,l}(f) := \frac{\partial f(P_{i,l})}{\partial g_{i,l}}, \quad \lambda^3_{i,l}(f) := \frac{\partial f(P_{i,l})}{\partial h_{i,l}}, \quad f \in C^1(S).$$

Then we define a nodal basis for $S^1_2(\Delta^{PS}_l)$ by the following interpolation problem: find functions $B^j_{i,l} \in S^1_2(\Delta^{PS}_l)$, $j = 1, 2, 3$, $i = 1, \ldots, N_l$, such that

$$\begin{aligned} \lambda^1_{k,l}(B^j_{i,l}) &= \rho^{-l}\delta_{j,1}\delta_{i,k}, \\ \lambda^2_{k,l}(B^j_{i,l}) &= \delta_{j,2}\delta_{i,k}, \\ \lambda^3_{k,l}(B^j_{i,l}) &= \delta_{j,3}\delta_{i,k}, \end{aligned} \tag{2.4}$$

for all $k = 1, \ldots, N_l$. Note that these basis functions satisfy $B^j_{i,l} \equiv (B^j_{i,l})_2|_S$, i.e., the spherical basis function $B^j_{i,l}$ is equal to the restriction of its homogeneous extension (2.2) of degree 2 to the sphere S. If we restrict $(B^j_{i,l})_2$ to the tangent plane touching S at $P_{i,l}$, we get the planar Hermite basis of [23] defined on this tangent plane. A detailed proof in a more general setting can be found in [18, Theorem 4.1].

We now show some stability properties of the nodal basis (2.4) that will be useful in the optimality proof of the BPX preconditioner.

LEMMA 2.3. *The nodal basis defined by (2.4) satisfies*

$$\|\sum_{i=1}^{N_l}\sum_{j=1}^{3} c^j_{i,l}B^j_{i,l}\|_{L_\infty} \sim \rho^{-l}\max_{i,j}|c^j_{i,l}|.$$

Proof. First we note that this result is well-known for the classical bivariate Hermite basis of Powell–Sabin type on planar triangulations. Indeed, the inequality $\gtrsim$ can be shown using the Markov inequality for polynomials ([8]), and the inequality $\lesssim$ can be deduced, for instance, from the work in [23, Section 6.2]. This result for the bivariate planar setting can be extended easily to the spherical setting exploiting the connection that we pointed out above. For a detailed proof, see [18, Corollary 4.2]. ∎

THEOREM 2.4. *If s is in $S^1_2(\Delta^{PS}_l)$, then for any $1 < p < \infty$ we have*

$$\|s\|^p_{L_p} \sim \rho^{-2l}\left(\sum_{i=1}^{N_l}|\lambda^1_{i,l}(s)|^p + \rho^{-lp}\sum_{i=1}^{N_l}\sum_{j=2}^{3}|\lambda^j_{i,l}(s)|^p\right).$$

Proof. Using the Markov inequality for spherical polynomials ([20, Prop. 4.3]), we infer that $|\lambda^j_{i,l}(s)| \lesssim \rho^l\|s\|_{L_\infty(\tau_i)}$ for $j = 2, 3$ with $\tau_i \in \Delta^{PS}_l$ such that $P_{i,l} \in \tau_i$. By mapping τ_i to a standard reference triangle and using the fact that all norms on the finite-dimensional space of polynomials are equivalent, we find that $\|s\|_{L_\infty(\tau_i)} \lesssim \rho^{2l/p}\|s\|_{L_p(\tau_i)}$, which implies

$$\rho^{-2l}\left(\sum_{i=1}^{N_l}|\lambda^1_{i,l}(s)|^p + \rho^{-lp}\sum_{i=1}^{N_l}\sum_{j=2}^{3}|\lambda^j_{i,l}(s)|^p\right) \lesssim \sum_{i=1}^{N_l}\sum_{j=1}^{3}\|s\|^p_{L_p(\tau_i)} \lesssim \|s\|^p_{L_p}.$$

The other inequality follows from the observation that

$$|s(v)|^p = \left| \sum_{i=1}^{N_l} \left(\rho^l \lambda_{i,l}^1(s) B_{i,l}^1(v) + \sum_{j=2}^{3} \lambda_{i,l}^j(s) B_{i,l}^j(v) \right) \right|^p$$

$$\lesssim \sum_{i=1}^{N_l} \left(\rho^{lp} |\lambda_{i,l}^1(s)|^p |B_{i,l}^1(v)|^p + \sum_{j=2}^{3} |\lambda_{i,l}^j(s)|^p |B_{i,l}^j(v)|^p \right),$$

which holds because at any $v \in S$ there are at most nine nonzero basis functions. We find that

$$\begin{aligned} \|s\|_{L_p}^p &\lesssim \sum_{i=1}^{N_l} \left(\rho^{lp} |\lambda_{i,l}^1(s)|^p \int_S |B_{i,l}^1(v)|^p dv + \sum_{j=2}^{3} |\lambda_{i,l}^j(s)|^p \int_S |B_{i,l}^j(v)|^p dv \right) \\ &\lesssim \rho^{-2l} \left(\sum_{i=1}^{N_l} |\lambda_{i,l}^1(s)|^p + \rho^{-lp} \sum_{i=1}^{N_l} \sum_{j=2}^{3} |\lambda_{i,l}^j(s)|^p \right), \end{aligned}$$

where we have used that $\|B_{i,l}^j\|_{L_\infty} \lesssim \rho^{-l}$ which follows from Lemma 2.3. ∎

3. Construction of the BPX preconditioners. In this section we construct BPX preconditioners for the problems (1.3) and (1.4), and we prove that these preconditioners are optimal. Let $k \in \{1, 2\}$. If k equals 1 we define S_l as the spline space $S_1^0(\Delta_l)$ and we solve problem (1.3). If k equals 2 we define S_l as the spline space $S_2^1(\Delta_l^{PS})$, leading to problem (1.4).

Let Q_l, $l = 0, 1, \ldots$, be a sequence of projectors on S_l which are orthogonal with respect to the inner product $(\cdot, \cdot)$ and let $Q_{-1} \equiv 0$. Let Ω be a subset of the sphere S and let $H^k(\Omega)$, $H^k(S)$ be the spherical Sobolev spaces as defined in [17, 20]. We prove the following theorem.

THEOREM 3.1. *Suppose $s \in S_n$. Then*

$$\|s\|_{H^k(S)}^2 \sim \sum_{l=0}^{n} \rho^{2kl} \, \|(Q_l - Q_{l-1})s\|_{L_2(S)}^2 . \tag{3.1}$$

Proof. Let Ω be a subset of S such that $\text{diam}(\Omega) \leq 1$. We define

$$R_\Omega \overline{v} := v := \frac{\overline{v}}{|\overline{v}|} \in S, \quad \overline{v} \in T_\Omega,$$

with T_Ω the tangent plane touching S at r_Ω, and r_Ω the center of a spherical cap of smallest possible radius containing Ω. Here a *spherical cap* is defined as the region of a sphere which lies on one side of a given plane that intersects with the sphere. Let $\overline{\Omega}$ be the image of Ω under R_Ω^{-1} and define $H^k(\overline{\Omega})$ as the usual Sobolev space on domains in $\mathbb{R}^2$. Let $(s)_k$ be the homogeneous extension (2.2) of degree k of s, and define $\overline{s}$ as the restriction of $(s)_k$ to $\overline{\Omega}$. The norm equivalence $\|s\|_{H^k(\Omega)} \sim \|\overline{s}\|_{H^k(\overline{\Omega})}$ holds, see Lemma 3.2 in [20]. Furthermore, we also have $\|s\|_{L_2(\Omega)} \sim \|\overline{s}\|_{L_2(\overline{\Omega})}$, see Lemma 3.1 in [20]. Now let $s = \sum_{l=0}^{n} s_l$ with each $s_l \in S_l$. Then it follows from the theory of homogeneous polynomials ([1, 2]) that $(s)_k = \sum_{l=0}^{n} (s_l)_k$, hence $\overline{s} = \sum_{l=0}^{n} \overline{s}_l$

with $\overline{s}_l$ the restriction of $(s_l)_k$ to $\overline{\Omega}$. Furthermore, each $\overline{s}_l$ is a member of the planar spline space $\overline{S}_l$ which is defined as $S_1^0(R_\Omega^{-1}(\Delta_l|_\Omega))$ for $k=1$ and as $S_2^1(R_\Omega^{-1}(\Delta_l^{PS}|_\Omega))$ for $k=2$. Proposition 2 in [21] claims that

$$\|\overline{s}\|_{H^k(\overline{\Omega})} \sim \inf \sum_{l=0}^{n} \rho^{2kl} \|\overline{s}_l\|^2_{L_2(\overline{\Omega})}$$

where the infimum must be taken with respect to all admissible representations $\sum_{l=0}^{n} \overline{s}_l$ of $\overline{s}$. From the norm equivalences above, we get

$$\|s\|_{H^k(\Omega)} \sim \inf \sum_{l=0}^{n} \rho^{2kl} \|s_l\|^2_{L_2(\Omega)}\,. \tag{3.2}$$

Now consider a finite collection of domains Ω_j with $\operatorname{diam}(\Omega_j) \le 1$, covering S. Equation (3.2) is valid for each sub-domain Ω_j. Furthermore, we have the equivalences $\|s_l\|^2_{L_2(S)} \sim \sum_j \|s_l\|^2_{L_2(\Omega_j)}$ and $\|s\|^2_{H^k(S)} \sim \sum_j \|s\|^2_{H^k(\Omega_j)}$. Hence,

$$\|s\|_{H^k(S)} \sim \inf \sum_{l=0}^{n} \rho^{2kl} \|s_l\|^2_{L_2(S)}\,,$$

which immediately implies (3.1), see [13, 21]. ☐

Remark 3.2. *Proposition 2 in [21] is formulated in terms of C^1 finite elements but it is clear that a similar result holds for C^0 finite elements (with obvious modifications).*

In view of (1.7) let us define the selfadjoint positive definite operator C_n^{-1} on S_n by

$$(C_n^{-1}u, v) = \sum_{l=0}^{n} \rho^{2kl}((Q_l - Q_{l-1})u, (Q_l - Q_{l-1})v), \tag{3.3}$$

and let A_n be the operator defined by (1.5) for $V = S_n$. By Poincaré's inequality on S, we have

$$a(u,u) \sim \|u\|^2_{H^k(S)}$$

under the constraint $\int_S u \, d\omega = 0$. Then Theorem 3.1 and (1.7) imply that

$$\kappa(C_n^{1/2} A_n C_n^{1/2}) = \mathcal{O}(1) \tag{3.4}$$

under the constraint that we fix the solution u of (1.3), (1.4) such that $\int_S u \, d\omega = 0$.

We now replace C_n by a spectrally equivalent and computationally simpler preconditioner $\widehat{C}_n$ given by

$$\widehat{C}_n := \sum_{l=0}^{n} \sum_{i=1}^{N_l} (\cdot, \phi_{i,l})\phi_{i,l} \tag{3.5}$$

for the problem (1.3) and by

$$\widehat{C}_n := \sum_{l=0}^{n} \sum_{i=1}^{N_l} \sum_{j=1}^{3} (\cdot, B^j_{i,l})B^j_{i,l} \tag{3.6}$$

for the problem (1.4). We say that two operators A and B are spectrally equivalent if

$$\frac{(Av,v)}{(v,v)} \sim \frac{(Bv,v)}{(v,v)}.$$

Let us focus on the biharmonic problem (1.4), i.e. take $k = 2$. By the orthogonality of the projectors Q_l one finds from (3.3) that

$$C_n = \sum_{l=0}^{n} \rho^{-4l}(Q_l - Q_{l-1}).$$

Because of the decaying scaling factors we are allowed to replace C_n by the spectrally equivalent operator

$$\widetilde{C}_n := \sum_{l=0}^{n} \rho^{-4l} Q_l.$$

From Theorem 2.4, we have the equivalence

$$\left\| \sum_{i=1}^{N_l} \sum_{j=1}^{3} c_{i,l}^j B_{i,l}^j \right\|_{L_2}^2 \sim \rho^{-4l} \sum_{i=1}^{N_l} \sum_{j=1}^{3} |c_{i,l}^j|^2,$$

and by the Riesz property this implies the existence of a dual basis $\{\widetilde{B}_{i,l}^j\}$ such that

$$\left\| \sum_{i=1}^{N_l} \sum_{j=1}^{3} c_{i,l}^j \widetilde{B}_{i,l}^j \right\|_{L_2}^2 \sim \rho^{4l} \sum_{i=1}^{N_l} \sum_{j=1}^{3} |c_{i,l}^j|^2.$$

The orthogonal projector Q_l has the representation

$$Q_l f = \sum_{i=1}^{N_l} \sum_{j=1}^{3} (f, B_{i,l}^j) \widetilde{B}_{i,l}^j.$$

Hence,

$$(Q_l f, f) = (Q_l f, Q_l f) = \|Q_l f\|_{L_2}^2 \sim \rho^{4l} \sum_{i=1}^{N_l} \sum_{j=1}^{3} |(f, B_{i,l}^j)|^2 = (\widehat{Q}_l f, f),$$

with

$$\widehat{Q}_l := \rho^{4l} \sum_{i=1}^{N_l} \sum_{j=1}^{3} (\cdot, B_{i,l}^j) B_{i,l}^j,$$

which shows that $\widehat{C}_n$ defined in (3.6) is spectrally equivalent to C_n such that, by (3.4),

$$\kappa(\widehat{C}_n^{1/2} A_n \widehat{C}_n^{1/2}) = \mathcal{O}(1).$$

The optimality of the preconditioner defined in (3.5) for problem (1.3) can be derived analogously using Theorem 2.2. We have, thus, proved the main result of this paper.

THEOREM 3.3. *The BPX preconditioners given by*

$$\sum_{l=0}^{n}\sum_{i=1}^{N_l}(\cdot,\phi_{i,l})\phi_{i,l} \quad \text{and} \quad \sum_{l=0}^{n}\sum_{i=1}^{N_l}\sum_{j=1}^{3}(\cdot,B^j_{i,l})B^j_{i,l}$$

yield uniformly bounded condition numbers for the problems (1.3), resp. (1.4).

COROLLARY 3.4. *Any basis of the general form in [18] which is stable in the sense of Theorem 2.4 gives rise to an optimal BPX preconditioner for (1.4).*

4. Numerical results. In this section we provide the results of numerical experiments illustrating the optimality of the BPX preconditioners developed in the earlier sections. We also compare the results of the BPX preconditioners with those obtained using the corresponding hierarchical preconditioners which are suboptimal.

The first problem that we solve is given by

$$-\Delta_S u = 2x \quad \text{on} \quad S, \tag{4.1}$$

and the exact solution u equals x, which can easily be checked since spherical harmonics are eigenfunctions of the Laplace–Beltrami operator on S ([19]). To discretize the problem (4.1) we use the basis functions $\phi_{i,l}$. We start from an almost uniform triangulation Δ_0 by projecting the twelve vertices of the regular icosahedron onto the sphere. These twelve points define a mesh consisting of twenty equal spherical triangles, cfr. [5]. The finer triangulatons Δ_l are constructed by subdividing the triangles of the previous coarser triangulation into four equal subtriangles (i.e. $\rho = 2$). Hence the dimension of the spline space increases like $2 + 10 \cdot 4^n$ with the refinement level n. Inner products of the form $(\nabla_S\phi_{i_1,l}, \nabla_S\phi_{i_2,l})$ will have to be computed. Hereto, we use a 3th order Gaussian quadrature formula on a triangle, see also [3, Prop. 4.1].

The second problem that we solve is given by

$$\Delta_S^2 u = 36xy \quad \text{on} \quad S, \tag{4.2}$$

and the exact solution u equals xy. In order to discretize (4.2) we have to compute inner products of the form $(\Delta_S B^{j_1}_{i_1,l}, \Delta_S B^{j_2}_{i_2,l})$. Since the basis functions $B^j_{i,l}$ are piecewise quadratic polynomials, we can use the formula

$$\Delta_S B^j_{i,l}(v) = \Delta B^j_{i,l}(v) - 6B^j_{i,l}(v), \quad v \in S,$$

with Δ the usual Laplace operator on $\mathbb{R}^3$, see [19]. Then, to evaluate the inner products, we use again a 3th order Gaussian quadrature formula on a triangle. We show results both for a dyadic and a triadic refinement procedure where we start from the same quasi-uniform triangulation Δ_0 as in the first problem (4.1). For the dyadic resp. triadic refinement procedure the dimension of the spline space increases like $6 + 30 \cdot 4^n$ resp. $6 + 30 \cdot 9^n$ with the refinement level n.

Note that the solution u in (4.1) and (4.2) is only unique up to a constant. From [1, Prop. 7.2] we find that constant functions on the sphere are contained in the spherical Powell–Sabin spline space $S^1_2(\Delta^{PS}_l)$ but not in the spherical piecewise linear spline space $S^0_1(\Delta_l)$. Hence, the stiffness matrix corresponding to the nodal basis $\{B^j_{i,l}\}$ will have one zero eigenvalue with an eigenvector corresponding to the constant function. The stiffness matrix corresponding to the nodal basis $\{\phi_{i,l}\}$ will have an eigenvalue of $\mathcal{O}(h^2)$ with an eigenvector that approximates the constant function up to discretization error $\mathcal{O}(h^2)$ w.r.t the L_2 norm. Recall that the condition numbers that we compute are given by $\kappa(C^{1/2}AC^{1/2}) = \lambda_{\max}/\lambda_{\min}$ where $\lambda_{\max}$ denotes the largest eigenvalue of $C^{1/2}AC^{1/2}$ and $\lambda_{\min}$ its smallest nonzero eigenvalue. Similarly, we also omit the smallest eigenvalue of $\mathcal{O}(h^2)$ for the Poisson equation. Note that, from Theorem 3.3, the BPX preconditioner uses all nodal basis functions on all levels. For each redundant basis function we will get a zero eigenvalue.

Tables 4.1, 4.2 and 4.3 show the results. We have used a nested iteration conjugate gradient method to solve the problem, i.e., by means of an outer iteration loop going from a coarse resolution level to the finest resolution level J we compute the solution to (4.1) or (4.2) at each level with the conjugate gradient method and we use the solution obtained at the previous coarser level as an initial guess. At each level we stop the conjugate gradient iteration if the H^k-norm of the residual is proportional to the discretization error which is of $\mathcal{O}(h)$. In [11] arguments are given for the fact that nested iteration is an asymptotically optimal method in the sense that it provides the solution u at the finest resolution level J up to discretization error in an overall amount of $\mathcal{O}(N_J)$ operations, provided that an optimal preconditioner is used.

	BPX			HB		
n	κ	residual	#iter	κ	residual	#iter
1	3.1	2.4897e-05	12	7.6	2.4974e-05	17
2	3.7	1.6766e-05	9	10.7	1.9546e-05	16
3	4.6	4.7350e-06	11	15.2	8.6198e-06	20
4	5.5	4.5474e-06	11	22.2	5.2361e-06	22
5	6.2	1.6705e-06	12	31.9	3.0622e-06	23
6	6.7	1.0193e-06	12	44.9	1.4750e-06	25
7	7.0	6.2720e-07	12	60.9	6.5043e-07	26
8	7.4	1.6451e-07	13	84.2	3.4960e-07	24

Table 4.1
Iteration history for problem (4.1).

	BPX			HB		
n	κ	residual	#iter	κ	residual	#iter
1	52.0	2.2290e-03	0	59.1	2.2222e-03	0
2	66.7	5.1424e-04	2	81.2	3.8158e-04	2
3	78.4	4.2928e-04	1	106.5	6.2316e-04	0
4	87.7	3.1846e-04	3	144.6	2.5778e-04	5
5	95.2	1.6570e-04	3	199.9	1.5944e-04	14
6	100.6	7.9261e-05	5	274.4	6.8452e-05	4
7	105.5	4.0583e-05	4	375.8	4.2450e-05	15

Table 4.2
Iteration history for problem (4.2), dyadic refinement.

	BPX			HB		
n	κ	residual	#iter	κ	residual	#iter
1	50.5	2.7160e-04	6	76.4	2.9346e-04	11
2	61.0	1.1074e-04	7	114.9	7.5921e-05	7
3	68.6	3.3405e-05	8	170.0	3.5187e-05	21
4	73.7	1.1717e-05	8	241.1	1.1716e-05	25

Table 4.3
Iteration history for problem (4.2), triadic refinement.

Each table has the same setup. The first column contains the resolution level n. Then we distinguish between the results for the BPX preconditioner and the results for the hierarchical basis (HB) preconditioner. For each preconditioner we display the spectral condition number κ of the system matrix for the linear system of equations that is solved. Moreover we show the H^k-norm of the residuals corresponding to the approximate solution, and the number of iterations that are needed on this level to reach discretization error accuracy.

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