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B-SPLINE-BASED MONOTONE MULTIGRID METHODS*

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Abstract. For the efficient numerical solution of elliptic variational inequalities on closed convex sets, multigrid methods based on piecewise linear finite elements have been investigated over the past decades. Essential to their success is the appropriate approximation of the constraint set on coarser grids which is based on function values for piecewise linear finite elements. On the other hand, there are a number of problems which profit from higher order approximations. Among these are the problem of pricing American options, formulated as a parabolic boundary value problem involving Black–Scholes' equation with a free boundary. In addition to computing the free boundary (the optimal exercise price of the option) of particular importance are accurate pointwise derivatives of the value of the stock option up to order two, the so-called Greek letters. In this paper, we propose a monotone multigrid method for discretizations in terms of B-splines of arbitrary order to solve elliptic variational inequalities on a closed convex set. In order to maintain monotonicity (upper bound) and quasi optimality (lower bound) of the coarse grid corrections, we propose an optimized coarse grid correction (OCGC) algorithm which is based on B-spline expansion coefficients. We prove that the OCGC algorithm is of optimal complexity of the degrees of freedom of the coarse grid and, therefore, the resulting monotone multigrid method is of asymptotically optimal multigrid complexity. Finally, the method is applied to a standard model for the valuation of American options. In particular, it is shown that a discretization based on B-splines of order four enables us to compute the second derivative of the value of the stock option to high precision.

Key words. variational inequality, linear complementary problem, monotone multigrid method, cardinal higher order B-spline, system of linear inequalities, optimized coarse grid correction algorithm, optimal complexity, convergence rates, American option, Greek letters, high precision

AMS subject classifications. 65M55, 35J85, 65N30, 65D07

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1. Introduction.

The motivation for this paper stems from an application in Mathematical Finance, the fair pricing of American options. In a standard model, this problem can be formulated as a parabolic boundary value problem involving Black–Scholes' equation [BS] with a free boundary. In addition to computing the free boundary (the optimal exercise price of the option), pointwise higher order derivatives of the solution (the value of the stock option) are particularly important. These so-called Greek letters are needed with high precision as they play a crucial role as hedge parameters in the analysis of market risks. Thus, a discretization in terms of higher order basis functions is preferable.

On the other hand, for the fast numerical solution of the resulting (semidiscrete) elliptic variational inequality, the method of choice is the monotone multigrid method developed in [Ko1, Ko2]. Multigrid methods have been proposed previously for such problems using second order discretizations (i.e., standard finite difference stencils or piecewise linear finite elements) in different variants [BC, HM, Ho, Ma] where, however, not all of them have assured, consequently, that the obstacle criterion is met. Using piecewise linear finite element ansatz functions, geometric considerations

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based on point values are used in [Ko1] to represent the problem-inherent obstacles on coarser grids in such a way that a violation of the obstacle is excluded. The difficulty to correctly identifying coarse grid approximations has also been the motivation for a cascadic multigrid algorithm for variational inequalities in [BBS] for which, however, no convergence theory is yet available.

In this paper, we generalize the monotone multigrid (MMG) method from [Ko1, Ko2] to discretizations involving higher order B-splines. One of the key ingredients of an MMG method are restrictions of the obstacle to coarser grids which satisfy the (upper) bound imposed by the obstacle (monotonicity) as well as a lower one which corresponds to the condition of quasi optimality in [Ko1]. We formulate the construction of coarse grid approximations as a linear constrained optimization problem with respect to the B-spline expansion coefficients. Our construction heavily profits from properties of B-splines [Bo, Sb]. In particular, we present with our optimized coarse grid correction (OCGC) algorithm a method to construct monotone and quasi-optimal coarse grid approximations to the obstacle function in optimal complexity of the coarse grid for B-spline basis functions of any degree.

Building the OCGC scheme into the MMG method, our higher order MMG method is shown to be of optimal multigrid complexity. Moreover, following the arguments in [Ko1], we can prove that our method is globally convergent and reduces asymptotically to a linear subspace correction method once the contact set has been identified [HzK]. Hence, we can expect particular robustness of the scheme and full multigrid efficiency in the asymptotic range in the numerical experiments. This is confirmed by computations for an American option pricing problem in terms of cubic B-splines. Details about the derivation of the problem of fair pricing American options and its formulation as a free boundary value problem and corresponding results can be found in [WHD, Hz]. Of course, once higher order MMG methods are available, they may be applied to other obstacle problems like Signorini's problem which has been solved using piecewise linear hat functions in [Kr].

This paper is structured as follows. In section 2 we introduce monotone multigrid methods, recollect the main features of B-splines, and specify a B-spline-based projected Gauss–Seidel relaxation as a smoothing component of the scheme. In section 3 the crucial ingredients of the higher order MMG schemes, suitable restriction operators for the obstacle function, are presented for B-spline functions of arbitrary degree in the univariate case. Their construction for higher spatial dimensions is presented in section 4 using tensor products. In section 5 some short remarks concerning the convergence theory for B-spline-based MMG schemes are made. Finally, in section 6 we present a numerical example of pricing American options. The convergence behavior of the projected Gauss–Seidel and the multigrid schemes is compared for basis functions of different orders. We conclude with an estimation of asymptotic multigrid convergence rates which exhibit full multigrid efficiency for the truncated version.

2. MMG methods.

2.1. Elliptic variational inequalities and linear complementary problems. Let Ω be a domain in $\mathbb{R}^d$ and $\mathcal{J}(v) := \frac{1}{2}a(v,v) - f(v)$ a quadratic functional induced by a continuous, symmetric, and H_0^1– elliptic bilinear form $a(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ and a linear functional $f : H_0^1(\Omega) \to \mathbb{R}$. As usual, $H_0^1(\Omega)$ is the subspace of functions belonging to the Sobolev space $H^1(\Omega)$ with zero trace on the boundary. We consider the constrained minimization problem

$$\text{find } u \in \mathcal{K} : \quad \mathcal{J}(u) \le \mathcal{J}(v) \quad \text{for all } v \in \mathcal{K} \tag{2.1}$$

on the closed and convex set

$$\mathcal{K} := \{v \in H_0^1(\Omega) : \ v(x) \leq g(x) \ \text{ for all } x \in \Omega\} \subset H_0^1(\Omega).$$

The function $g \in H_0^1(\Omega)$ represents an upper obstacle for the solution $u \in H_0^1(\Omega)$. Lower obstacles can be treated in the obvious analogous way. If g satisfies $g(x) \geq 0$ for all $x \in \partial\Omega$, then problem (2.1) admits a unique solution $u \in \mathcal{K}$ by the Lax–Milgram theorem. It is well-known that (2.1) can be rewritten as a variational inequality; see, e.g., [EO, KS]: find $u \in \mathcal{K} : \ a(u, v-u) \geq f(v-u)$ for all $v \in \mathcal{K}$ or, equivalently, as a *linear complementary problem*

$$\begin{aligned} \mathcal{L}u &\geq f, \\ u &\leq g, \\ (u-g)(\mathcal{L}u - f) &= 0 \end{aligned} \tag{2.2}$$

almost everywhere in Ω. Here $\mathcal{L} : H_0^1(\Omega) \to H^{-1}(= (H_0^1(\Omega))')$ is the Riesz operator defined by $\langle \mathcal{L}u, v\rangle := a(u,v)$ for all $v \in H_0^1(\Omega)$.

Discretizing in a finite dimensional spline space S_L of piecewise polynomials on a grid Δ_L with uniform grid spacing h_L leads to the discrete formulation of (2.1),

$$\text{find } \ u_L \in \mathcal{K}_L : \quad \mathcal{J}(u_L) \leq \mathcal{J}(v_L) \qquad \text{for all } \ v_L \in \mathcal{K}_L \tag{2.3}$$

on the closed and convex set $\mathcal{K}_L := \{v_L \in S_L : \ v_L(x) \leq g_L(x) \ \text{ for all } x \in \Omega\} \subset S_L$, or, equivalently,

$$\begin{aligned} \mathcal{L}_L u_L &\geq f_L, \\ u_L &\leq g_L, \\ (u_L - g_L)(\mathcal{L}_L u_L - f_L) &= 0. \end{aligned} \tag{2.4}$$

In [BHR] regularity $u \in H^{5/2-\epsilon}(\Omega)$ of the solution u to (2.2) is shown for arbitrary $\epsilon > 0$. Moreover, error estimates $\|u - u_L\|_{H^1(\Omega)} = O(h_L)$ and $\|u - u_L\|_{H^1(\Omega)} = O(h_L^{3/2-\epsilon})$ are proved in the case of piecewise linear (respectively, piecewise quadratic) functions, provided the functions f, g are sufficiently regular.

2.2. The MMG-algorithm. For solving (2.3) numerically, a now-popular method is the MMG method [Ko1]. By adding a projection step and employing specific restriction operators, it can be implemented as a variant of a standard multigrid scheme. Let $S_1 \subset S_2 \subset \cdots \subset S_L \subset H_0^1(\Omega)$ be a nested sequence of finite dimensional spaces, and let $u_L^\nu \in S_L$ be the approximation in the νth iteration of the MMG method. The basic multigrid idea is that the error $v_L := u_L - u_L^{\nu,1}$ between the smoothed iterate $u_L^{\nu,1} := \mathcal{S}(u_L^\nu)$ ($\mathcal{S}$ always being the standard Gauss–Seidel iteration) and the exact solution u_L can be approximated without essential loss of information on a coarser grid Δ_{L-1}. We explain how this is realized in the case of a linear complementary problem for two grids Δ_L and Δ_{L-1}. Introducing the defect $d_L := f_L - \mathcal{L}_L u_L^{\nu,1}$, (2.4) can be written as

$$\begin{aligned} \mathcal{L}_L v_L &\geq d_L, \\ v_L &\leq g_L - u_L^{\nu,1}, \\ (v_L - g_L + u_L^{\nu,1})(\mathcal{L}_L v_L - d_L) &= 0. \end{aligned} \tag{2.5}$$

On a coarser grid Δ_{L-1}, the defect problem can now be approximated by

$$\begin{aligned} \mathcal{L}_{L-1} v_{L-1} &\geq d_{L-1}, \\ v_{L-1} &\leq g_{L-1}, \\ (v_{L-1} - g_{L-1})(\mathcal{L}_{L-1} v_{L-1} - d_{L-1}) &= 0, \end{aligned}$$

where $d_{L-1} := r\, d_L$ and $g_{L-1} := \tilde{r}(g_L - u_L^{\nu,1})$ with (different) restriction operators $r, \tilde{r} : S_L \to S_{L-1}$. The solution v_{L-1} of the coarse grid problem is then used as an approximation to the error v_L. It is first transported back to the fine grid by a prolongation operator p and is then added to the approximation $u_L^{\nu,1}$. It is important that the restriction $\tilde{r}$ is chosen such that the new iterate satisfies the constraint

$$u_L^{\nu,2} := u_L^{\nu,1} + p v_{L-1} \leq g_L \tag{2.6}$$

on the fine grid. Applying this idea recursively on several different grids, one obtains the MMG method for linear complementary problems.

ALGORITHM 2.1. **MMG**$_\ell$ *(νth cycle on level $\ell \geq 1$).*
Let $u_\ell^\nu \in S_\ell$ be a given approximation.

1. *A priori smoothing and projection:* $u_\ell^{\nu,1} := (\mathcal{P} \circ \mathcal{S}(u_\ell^\nu))^{\eta_1}$.
2. *Coarse grid correction:* $d_{\ell-1} := r(f_\ell - \mathcal{L}_\ell u_\ell^{\nu,1})$,
   $g_{\ell-1} := \tilde{r}(g_\ell - u_\ell^{\nu,1})$,
   $\mathcal{L}_{\ell-1} := r \mathcal{L}_\ell p$.

   If $\ell = 1$, solve exactly the linear complementary problem

$$\begin{aligned} \mathcal{L}_{\ell-1} v &\geq d_{\ell-1}, \\ v &\leq g_{\ell-1}, \\ (v - g_{\ell-1})(\mathcal{L}_{\ell-1} v - d_{\ell-1}) &= 0, \end{aligned}$$

   and set $v_{\ell-1} := v$.
   If $\ell > 1$, do γ steps of **MMG**$_{\ell-1}$ *with initial value $u_{\ell-1}^0 := 0$ and solution $v_{\ell-1}$.*
   Set $u_\ell^{\nu,2} := u_\ell^{\nu,1} + p v_{\ell-1}$.
3. *A posteriori smoothing and projection:* $u_\ell^{\nu,3} := (\mathcal{P} \circ \mathcal{S}(u_\ell^{\nu,2}))^{\eta_2}$.

   Set $u_\ell^{\nu+1} := u_\ell^{\nu,3}$.

The number of a priori and a posteriori smoothing steps is denoted by η_1 and η_2, respectively. For $\gamma = 1$ one obtains a V-cycle, for $\gamma = 2$ a W-cycle. $\mathcal{P}$ denotes a projection operator defined in (2.7) and (2.11).

Condition (2.6) leads to an inner approximation of the solution set $\mathcal{K}_L$ and ensures that the multigrid scheme is robust [Ko1]. Striving for optimal multigrid efficiency, satisfaction of the constraint should *not* be checked by interpolating v_ℓ back to the finest grid. Instead, special restriction operators $\tilde{r}$ are needed for the obstacle function. A corresponding construction for B-splines of general order k will be introduced in sections 3 and 4. Next we discuss the projection step for general order B-splines.

2.3. A B-spline-based projected Gauss–Seidel scheme. Since the operator $\mathcal{L}$ is symmetric positive definite and continuous piecewise linear functions are used for discretization, the discrete form (2.4) can be solved by the projected Gauss–Seidel scheme; see, e.g., [Cr]. Given an iterate u_L^ν, a standard Gauss–Seidel sweep $\bar{u}_L^\nu := \mathcal{S}\,(u_L^\nu)$ is supplemented by a projection $u_L^{\nu+1} = \mathcal{P}\,\bar{u}_L^\nu$ into the convex set $\mathcal{K}_L$. If

S_L consists of hat functions, the projection can be defined for given grid points $\{\theta_i\}_i$ by

$$\mathcal{P}\, v_L(\theta_i) := \min\{v_L(\theta_i),\, g_L(\theta_i)\}. \tag{2.7}$$

For higher order functions v_L, the difficulty arises already in the univariate case that for given $x \in [\theta_i, \theta_{i+1}]$ the estimate

$$\min\{v_L(\theta_i), v_L(\theta_{i+1})\} \le v_L(x) \le \max\{v_L(\theta_i), v_L(\theta_{i+1})\} \tag{2.8}$$

is no longer valid. Thus, controlling function values on grid points is not a sufficient criterion in this case. Instead, we propose here a construction using higher order B-splines, which compares B-spline expansion coefficients instead of function values and heavily profits from the fact that B-splines are nonnegative. We begin with the univariate case. For readers' convenience, we recall the relevant facts about B-spline bases from [Bo].

DEFINITION 2.2 (B-spline basis functions). *For $k \in \mathbb{N}$ and $n \in \mathbb{N}$ let $T := \{\theta_i\}_{i=1,\dots,n+k}$ be an* expanded knot sequence *with uniform grid spacing h_L in the interior of the interval $I := [a, b]$ of the form*

$$\theta_1 = \cdots = \theta_k = a < \theta_{k+1} < \cdots < \theta_n < b = \theta_{n+1} = \cdots = \theta_{n+k}. \tag{2.9}$$

Then the B-spline basis functions $N_{i,k}$ of order k are recursively defined for $i = 1, \dots, n$ by

$$\begin{aligned} N_{i,1}(x) &= \begin{cases} 1 & \text{if } x \in [\theta_i, \theta_{i+1}) \\ 0 & \text{else,} \end{cases} \\ N_{i,k}(x) &= \frac{x - \theta_i}{\theta_{i+k-1} - \theta_i}\, N_{i,k-1}(x) + \frac{\theta_{i+k} - x}{\theta_{i+k} - \theta_{i+1}}\, N_{i+1,k-1}(x) \end{aligned} \tag{2.10}$$

for $x \in I$.

It is known that $\operatorname{supp} N_{i,k} \subseteq [\theta_i,\, \theta_{i+k}]$ (local support), $N_{i,k}(x) \ge 0$ for all $x \in I$ (nonnegativity), and $N_{i,k} \in C^{k-2}(I)$ (differentiability) holds. Moreover, the set $\Sigma_L := \{N_{1,k}, \dots, N_{n,k}\}$ constitutes a locally independent and unconditionally stable basis with respect to $\|\cdot\|_{L_p}$, $1 \le p \le \infty$ for the finite dimensional space $S_L = \mathcal{N}_{k,T} := \operatorname{span} \Sigma_L$ of the splines of order k.

LEMMA 2.3. *If the B-spline coefficients of $v_L, g_L \in \mathcal{N}_{k,T} = S_L$ satisfy $v_i \le g_i$ for all $i = 1, \dots, n$, then $v_L(x) \le g_L(x)$ holds for all $x \in I$.*

Proof. Using the representation $v_L = \sum_{i=1}^n v_i\, N_{i,k}$ and $g_L = \sum_{i=1}^n g_i\, N_{i,k}$ and the nonnegativity $N_{i,k}(x) \ge 0$ for all $x \in I$, we deduce that $g_L(x) - v_L(x) = \sum_{i=1}^n (g_i - v_i)\, N_{i,k}(x) \ge 0$ for all $x \in I$. □

Here and in section 5, we use the subscript i in $v_i = (v_L)_i$ to denote B-spline expansion coefficients.

The projection can now be defined for B-spline functions of general order k similar to (2.7), but now involving expansion coefficients by setting

$$\mathcal{P}\, v_i := \min\{v_i,\, g_i\}. \tag{2.11}$$

Using the same arguments as in [Cr], the resulting projected Gauss–Seidel scheme still converges since the discrete solution set $\{\mathbf{v} \in \mathbb{R}^n : v_i \le g_i \text{ for } i = 1, \dots, n\}$ describes a cuboid in $\mathbb{R}^n$. Moreover, if the problem is nondegenerate, the contact set, defined by all coefficients for which equality holds, is identified after a finite number of iterations [Cr, EO].

We treat the multivariate case by taking tensor products. Specifying the domain Ω as $\Omega := \prod_{\ell=1}^{d}[a_\ell, b_\ell] \subset \mathbb{R}^d$, the ith d-dimensional tensor product B-spline of order k on a tensorized extended knot sequence $T^{(d)}$ is defined by

$$N_{i,k}^{(d)}(x) := \prod_{\ell=1}^{d} N_{i_\ell,k}(x_\ell), \qquad x \in \Omega, \tag{2.12}$$

where $i := (i_1, \ldots, i_d)$ denotes a multi-index. Defining S_L in analogy to the univariate case, the result of Lemma 2.3 immediately carries over to the d-dimensional setting.

3. Construction of monotone and quasi-optimal obstacle approximations. In this section, the second essential ingredient for our B-spline-based MMG methods is provided, the construction of so-called *monotone* and *quasi-optimal coarse grid approximations* of the obstacle function, which lead to suitable restriction operators $\tilde{r}$. We begin with the univariate case; the extension to d dimensions follows in section 4. We consider in what follows only two grids, as the generalization to several grids is obvious. Given an obstacle function $\tilde{S}$ which is defined on a fine grid $\Delta \subset I$, we provide an approximation S with respect to a coarser grid T which satisfies

1. $S(x) \leq \tilde{S}(x)$ for all $x \in I$;
2. $S(x) \geq L_k(x)$ for all $x \in I$ and a still-to-be-specified lower barrier $L_k(x)$ provided in section 3.2;
3. $S \approx \tilde{S}$ with respect to a target functional F_k defined below in (3.10).

The first condition ensures the monotonicity and robustness of the multigrid scheme, the second an asymptotical reduction of the method to a linear relaxation, and the third an efficient coarse grid correction. As the construction is used as a component of the monotone multigrid scheme striving for optimal computational multigrid complexity, it also has to satisfy

4. the number of arithmetic operations must be of order $O(n)$, where n denotes the number of degrees of freedom on the coarse grid.

Specifically, let T be an extended knot sequence with grid spacing H as in (2.9) and let $\Delta := \{\tilde{\theta}_i\}_{i=1,\ldots,\tilde{n}+k}$ be a *finer* knot sequence

$$\tilde{\theta}_1 = \cdots = \tilde{\theta}_k = a < \tilde{\theta}_{k+1} < \cdots < \tilde{\theta}_{\tilde{n}} < b = \tilde{\theta}_{\tilde{n}+1} = \cdots = \tilde{\theta}_{\tilde{n}+k} \tag{3.1}$$

with grid spacing $h = \frac{1}{2}H$. It is defined such that $\theta_i = \tilde{\theta}_{2i-k}$ for $i = k, \ldots, n+1$ and $\frac{1}{2}(\theta_{i-1} + \theta_i) = \tilde{\theta}_{2i-k-1}$ for $i = k+1, \ldots, n+1$. Then it holds that

$$\tilde{n} = 2n + 1 - k. \tag{3.2}$$

The corresponding spline spaces are $\mathcal{N}_{k,\Delta}$ and $\mathcal{N}_{k,T}$ with member functions $N_{i,k,\Delta}$ and $N_{i,k,T}$, respectively. Now let the obstacle function on the fine grid $\tilde{S} \in \mathcal{N}_{k,\Delta}$ and its approximation $S \in \mathcal{N}_{k,T}$ be expanded as

$$\tilde{S} = \sum_{i=1}^{\tilde{n}} \tilde{c}_i \, N_{i,k,\Delta} =: \tilde{\mathbf{c}}^T \, \mathbf{N}_{k,\Delta}, \qquad S = \sum_{i=1}^{n} c_i \, N_{i,k,T} =: \mathbf{c}^T \, \mathbf{N}_{k,T}. \tag{3.3}$$

There is a natural *prolongation operator* p from $\mathcal{N}_{k,T}$ to $\mathcal{N}_{k,\Delta}$ for B-splines $N_{i,k,T}$ in terms of their refinement or mask coefficients [Bo, Sb]. In the special case $H = 2\,h$

considered here the refinement relation is given by

$$N_{i,k,T} = \sum_{j=0}^{k} a_j \, N_{2i-k+j,k,\Delta} \tag{3.4}$$

with the *subdivision* or *mask* coefficients

$$a_j := 2^{1-k} \binom{k}{j} \qquad \text{for } j = 0, \ldots, k. \tag{3.5}$$

In step 2 of Algorithm 2.1, we choose the restriction r as the adjoint of p, following [Ha]. However, for the obstacle function the restriction operator r cannot be used since it does not satisfy condition (2.6).

3.1. Monotone coarse grid approximations. There is a vast amount of literature, see, e.g., [DV, Mv, Pi] especially from approximation theory, dealing with monotone approximations to a given function g. The function $\hat{g}$ is a *monotone* (or *one-sided*) *lower approximation* to g if $\hat{g}(x) \leq g(x)$ for all $x \in I$. There the number n of degrees of freedom of the function $\hat{g}$ is chosen such that a given approximation accuracy can be reached. In contrast to these studies, the question here is different, since the number n of degrees of freedom is given by the mesh size H.

DEFINITION 3.1 (monotone coarse grid approximation). *For knot sequences T and Δ from* (2.9) *and* (3.1), *respectively, we call $S \in \mathcal{N}_{k,T}$ a* monotone lower coarse grid approximation *to $\tilde{S} \in \mathcal{N}_{k,\Delta}$ if $S(x) \leq \tilde{S}(x)$ holds for all $x \in I$.*

For hat functions such approximations are constructed in [Ma, Ko1]. A corresponding construction for higher order functions has, to our knowledge, not been provided so far. In view of Lemma 2.3 we propose here to control B-spline expansion coefficients.

THEOREM 3.2 (monotone coarse grid approximation). *Let $\tilde{S} \in \mathcal{N}_{k,\Delta}$ be an upper obstacle with $\tilde{S} = \tilde{\mathbf{c}}^T \mathbf{N}_{k,\Delta}$ for a given order k and the knot sequence Δ from* (3.1). *Then $S \in \mathcal{N}_{k,T}$ with $S = \mathbf{c}^T \mathbf{N}_{k,T}$ defined on the knot sequence T from* (2.9) *is a monotone lower coarse grid approximation to $\tilde{S}$ if the inequality system*

$$A_k \, \mathbf{c} \leq \tilde{\mathbf{c}} \tag{3.6}$$

is satisfied. The two-slanted *matrix A_k is defined by*

$$A_k := \begin{pmatrix} a_{k-1} & a_{k-3} & \ddots & & & \\ a_k & a_{k-2} & \ddots & & & \\ & a_{k-1} & & a_0 & & \\ & a_k & & a_1 & & \\ & & \ddots & a_2 & & \\ & & & \vdots & & \\ & & & a_{k-1} & \ddots & \\ & & & a_k & & \\ & & & & \ddots & a_0 \\ & & & & & a_1 \end{pmatrix} \in \mathbb{R}^{\tilde{n} \times n}$$

with the subdivision coefficients a_j from (3.5) *and has maximal rank.*

Proof. The proof relies on the subdivision property (3.4) and on the nonnegativity of B-splines. We only consider the case k even as the other case is analogous. Substituting (3.4) into (3.3) and sorting according to the basis functions $N_{i,k,\Delta}$ leads to

$$S(x) = \sum_{\substack{i=1 \\ i \text{ odd}}}^{\tilde{n}} \left(a_{k-1}\, c_{(i+1)/2} + a_{k-3}\, c_{(i+3)/2} + \cdots + a_1\, c_{(i+k-1)/2}\right) N_{i,k,\Delta}(x)$$
$$+ \sum_{\substack{i=2 \\ i \text{ even}}}^{\tilde{n}-1} \left(a_k\, c_{i/2} + a_{k-2}\, c_{(i+2)/2} + \cdots + a_0\, c_{(i+k)/2}\right) N_{i,k,\Delta}(x),$$

where all c_j with $j < 1$ or $j > n$ are treated as zero. Defining the coefficients

$$d_i := \begin{cases} \tilde{c}_i - \left(a_{k-1}\, c_{(i+1)/2} + a_{k-3}\, c_{(i+3)/2} + \cdots + a_1\, c_{(i+k-1)/2}\right) & \text{if } i \text{ is odd,} \\ \tilde{c}_i - \left(a_k\, c_{i/2} + a_{k-2}\, c_{(i+2)/2} + \cdots + a_0\, c_{(i+k)/2}\right) & \text{if } i \text{ is even,} \end{cases}$$

which can be written in compact matrix/vector form as

$$d_i = \tilde{c}_i - \left(A_k\, \mathbf{c}\right)_i \tag{3.7}$$

(involving the ith component of the vector $A_k\, \mathbf{c}$), we obtain

$$\tilde{S}(x) - S(x) = \sum_{i=1}^{\tilde{n}} d_i\, N_{i,k,\Delta}(x). \tag{3.8}$$

By Lemma 2.3 we have $\tilde{S}(x) - S(x) \geq 0$ for all $x \in I$, provided $d_i \geq 0$ holds for all $i = 1, \ldots, \tilde{n}$. By (3.7), we obtain the inequality system (3.6). Since the B-splines form bases for $\mathcal{N}_{k,T}$ and $\mathcal{N}_{k,\Delta}$, the matrix A_k has full rank for each k. ∎

Example 3.3. In the special case of continuous, piecewise linear functions ($k = 2$), C^1-smooth; piecewise quadratic ($k = 3$); and C^2-smooth, piecewise cubic ($k = 4$) splines, one has

$$A_2 = \begin{pmatrix} 1 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ & 1 & & & \\ & \frac{1}{2} & \frac{1}{2} & & \\ & \ddots & & \frac{1}{2} & \\ & & & 1 & \\ & & & \frac{1}{2} & \frac{1}{2} \\ & & & & 1 \end{pmatrix} \in \mathbb{R}^{(2n-1)\times n}, \quad A_3 = \frac{1}{4} \begin{pmatrix} 3 & 1 & & & & \\ 1 & 3 & & & & \\ & 3 & 1 & & & \\ & 1 & 3 & & & \\ & \ddots & & \ddots & & \\ & & & 3 & 1 & \\ & & & 1 & 3 & \\ & & & & 3 & 1 \\ & & & & 1 & 3 \end{pmatrix} \in \mathbb{R}^{(2n-2)\times n},$$

$$A_4 = \frac{1}{8} \begin{pmatrix} 4 & 4 & & & & \\ 1 & 6 & 1 & & & \\ & 4 & 4 & & & \\ & 1 & 6 & 1 & & \\ & \ddots & & \ddots & & \\ & & 1 & 6 & 1 & \\ & & & 4 & 4 & \\ & & & 1 & 6 & 1 \\ & & & & 4 & 4 \end{pmatrix} \in \mathbb{R}^{(2n-3)\times n}.$$

TABLE 3.1
The values β_k and γ_k for orders $k = 2, 4, 6, 8$.

k	2	4	6	8
β_k	1	$\frac{2}{3}$	$\frac{17}{30}$	$\frac{166}{315}$
γ_k	0	$\frac{1}{3}$	$\frac{13}{30}$	$\frac{149}{315}$

3.2. Quasi-optimal coarse grid approximations. Now we can immediately derive a monotone lower coarse approximation.

PROPOSITION 3.4. *The spline $L_k := \mathbf{q}^T \mathbf{N}_{k,T} \in \mathcal{N}_{k,T}$ with coefficients*

$$q_i := \min\{\tilde{c}_{2i-k}, \ldots, \tilde{c}_{2i}\} \quad \text{for } i = 1, \ldots, n \tag{3.9}$$

(leaving out $\tilde{c}_j$ in the right-hand side if $j < 1$ or $j > \tilde{n}$) is a monotone lower coarse grid approximation to $\tilde{S} = \tilde{\mathbf{c}}^T \mathbf{N}_{k,\Delta} \in \mathcal{N}_{k,\Delta}$.

Proof. As all row sums of A_k are equal to one, the vector $\mathbf{q} := (q_1, \ldots, q_n)^T$ defined in (3.9) obviously satisfies the inequality system $A_k\,\mathbf{q} \le \tilde{\mathbf{c}}$ so that the assertion directly follows from Theorem 3.2. ∎

Remark 3.5. In the special case $k = 2$, the restriction operator $\hat{r}:\ \mathcal{N}_{2,\Delta} \to \mathcal{N}_{2,T}$, $\tilde{S} \mapsto L_2$ induced by Proposition 3.4 coincides with the restriction operator from [Ma].

As is illustrated in Figures 3.1 and 3.2 for the cases $k = 2$ and $k = 3$, the approximation L_k can be further improved in many cases. This will be the subject of the next subsections: there $\mathbf{q}$ is interpreted as a componentwise lower barrier for the B-spline coefficients $\mathbf{c}$ of the desired coarse grid approximation.

DEFINITION 3.6 (quasi-optimal coarse grid approximation). *We call a monotone lower coarse grid approximation $S = \mathbf{c}^T \mathbf{N}_{k,T}$ to the spline $\tilde{S} = \tilde{\mathbf{c}}^T \mathbf{N}_{k,\Delta}$* quasi-optimal *if it is an improvement over L_k in the sense that $\mathbf{c} \ge \mathbf{q}$ holds with $\mathbf{q}$ defined in* (3.9).

3.3. A linear optimization problem. Aiming at improving the coarse grid approximation L_k from Proposition 3.4, we define an *optimal* monotone and quasi-optimal coarse grid approximation $S = \mathbf{c}^T \mathbf{N}_{k,T}$ to a given $\tilde{S} = \tilde{\mathbf{c}}^T \mathbf{N}_{k,\Delta}$ by formulating a linear optimization problem. We choose a target functional F_k which estimates the sum of the distances from approximation to obstacle on all coarse grid points, i.e.,

$$F_k(\mathbf{c}) := \sum_{\theta \in T} |\tilde{S}(\theta) - S(\theta)|. \tag{3.10}$$

LEMMA 3.7. *The function F_k defined in* (3.10) *is a linear function $\mathbb{R}^n \to \mathbb{R}$ of the form*

$$F_k(\mathbf{c}) = \boldsymbol{\xi}^T \mathbf{c} + \eta, \tag{3.11}$$

where

$$\boldsymbol{\xi} := -A_k^T \mathbf{s}_k \in \mathbb{R}^n, \quad \mathbf{s}_k := (\beta_k, \gamma_k, \beta_k, \ldots)^T \in \mathbb{R}^{\tilde{n}}, \ \text{and } \eta := \mathbf{s}_k^T \tilde{\mathbf{c}} \in \mathbb{R}. \tag{3.12}$$

The values β_k and γ_k can be computed explicitly: for odd k we have $\beta_k = \gamma_k = \frac{1}{2}$, and for even $k = 2, 4, 6, 8$ the values are displayed in Table 3.1.

Proof. By Theorem 3.2 we have $|\tilde{S}(x) - S(x)| = \tilde{S}(x) - S(x)$ for all $x \in I$. Using (3.8) we obtain

$$F_k(\mathbf{c}) = \sum_{\theta \in T} \left(\tilde{S}(\theta) - S(\theta)\right) = \sum_{\theta \in T} \sum_{i=1}^{\tilde{n}} d_i\, N_{i,k,\Delta}(\theta) = \sum_{i=1}^{\tilde{n}} d_i \sum_{\theta \in T} N_{i,k,\Delta}(\theta). \tag{3.13}$$

Abbreviating $(\tilde{\mathbf{s}}_k)_i := \sum_{\theta\in T} N_{i,k,\Delta}(\theta)$, we next show that $\tilde{\mathbf{s}}_k$ coincides with $\mathbf{s}_k$ defined in (3.12). In fact, $\sum_{\theta\in\Delta} N_{i,k,\Delta}(\theta) = 1$ is easily shown by induction for $k \in \mathbb{N}$. For odd k we can use a simple symmetry argument to conclude $(\tilde{\mathbf{s}}_k)_i = \frac{1}{2}$. For even k two cases must be distinguished according to the position of $N_{i,k,\Delta}$. Evaluating the B-spline on coarse grid points leads to $(\tilde{\mathbf{s}}_k)_i = \beta_k$ if $\theta_{i+k/2} \in T$ and $(\tilde{\mathbf{s}}_k)_i = \gamma_k$ in the other case. For orders $k = 2, 4, 6, 8$, the concrete values β_k and γ_k are displayed in Table 3.1. Thus, we have $(\mathbf{s}_k)_i = (\tilde{\mathbf{s}}_k)_i$ and employing (3.7) in (3.13) leads to (3.11), i.e., $F_k(\mathbf{c}) = \sum_{i=1}^{\tilde{n}} (\mathbf{s}_k)_i\,(\tilde{c}_i - (A_k\,\mathbf{c})_i) = \mathbf{s}_k^T\tilde{\mathbf{c}} - \mathbf{s}_k^T\,A_k\,\mathbf{c} = \boldsymbol{\xi}^T\mathbf{c} + \eta$. ∎

We can now define an optimal monotone and quasi-optimal coarse grid approximation as the solution of the linear optimization problem

$$\text{(3.14)}\qquad \begin{aligned} &\text{Minimize the target functional} && F_k(\mathbf{c}) = \boldsymbol{\xi}^T\mathbf{c} + \eta \\ &\text{with respect to the constraints} && A_k\,\mathbf{c} \le \tilde{\mathbf{c}} \quad\text{and}\quad \mathbf{c} \ge \mathbf{q}. \end{aligned}$$

Here $A_k \in \mathbb{R}^{\tilde{n}\times n}$, $\tilde{\mathbf{c}} \in \mathbb{R}^{\tilde{n}}$, and $\mathbf{q} \in \mathbb{R}^n$ are defined as before with $\tilde{n} = 2n - k + 1$ and $\boldsymbol{\xi} \in \mathbb{R}^n$ and $\eta \in \mathbb{R}$ are given as in (3.12). The upper inequality guarantees the monotonicity of the approximation by Theorem 3.2, while the second one ensures quasi optimality by Proposition 3.4.

3.4. Solution of the linear optimization problem. Via the linear optimization formulation (3.14) a (with respect to the target functional F_k) optimal monotone and quasi-optimal coarse grid approximation may now be obtained, in principle, by the *simplex algorithm*; see, e.g., [Sj]. Here the point $\mathbf{q} \in \mathbb{R}^n$ could be used as a starting corner by Proposition 3.4. In a multigrid scheme, however, the simplex algorithm should not be used because the optimal complexity $O(n)$ would be destroyed. As shown next, a direct solution for $k = 2$ can be obtained by the *Fourier–Motzkin elimination*; see, e.g., [Sj]. For the general case $k > 2$ we present afterwards an approximate solution algorithm which can be applied in optimal complexity.

LEMMA 3.8 (direct solution for hat functions). *For $k = 2$ and given $\tilde{\mathbf{c}} \in \mathbb{R}^{\tilde{n}}$, the solution of the linear optimization problem* (3.14) *is recursively given by*

$$\text{(3.15)}\qquad \begin{aligned} c_1 &:= \min\{\tilde{c}_1,\, 2\tilde{c}_2 - q_2\} \\ c_i &:= \min\{2\,\tilde{c}_{2i-2} - c_{i-1},\, \tilde{c}_{2i-1},\, 2\,\tilde{c}_{2i} - q_{i+1}\} \quad \textit{for } i = 2,\dots,n-1, \\ c_n &:= \min\{2\,\tilde{c}_{2n-2} - c_{n-1},\, \tilde{c}_{2n-1}\} \end{aligned}$$

with $q_i = \min\{\tilde{c}_{2i-2},\, \tilde{c}_{2i-1},\, \tilde{c}_{2i}\}$ for $i = 1,\dots,n$ defined in (3.9). *In particular, $S = \mathbf{c}^T\,\mathbf{N}_{k,T}$ is a monotone and quasi-optimal coarse grid approximation to the obstacle $\tilde{S} = \tilde{\mathbf{c}}^T\,\mathbf{N}_{2,\Delta}$.*

Proof. First, the n conditions $-\mathbf{c} \le -\mathbf{q}$ are integrated into the inequality system $A_2\,\mathbf{c} \le \tilde{\mathbf{c}}$ from Theorem 3.2. Then, Fourier–Motzkin elimination is applied to the resulting $(3n-1)\times n$ inequality system so that we obtain the solution range

$$\begin{aligned} q_1 &\le c_1 \le \min\{\tilde{c}_1,\, 2\tilde{c}_2 - q_2\}, \\ q_i &\le c_i \le \min\{2\tilde{c}_{2i-2} - c_{i-1},\, \tilde{c}_{2i-1},\, 2\tilde{c}_{2i} - q_{i+1}\} \quad \text{for } i = 2,\dots,n-1, \\ q_n &\le c_n \le \min\{2\tilde{c}_{2n-2} - c_{n-1},\, \tilde{c}_{2n-1}\}. \end{aligned}$$

Because of (3.9), $q_1 \le \min\{\tilde{c}_1,\, 2\tilde{c}_2 - q_2\}$ holds. To minimize the target function F_2 given by Lemma 3.7, all coefficients c_i must be chosen as large as possible which leads to (3.15). ∎

Remark 3.9. The restriction operator $\tilde{r} : \mathcal{N}_{2,\Delta} \to \mathcal{N}_{2,T}$, $\tilde{S} \mapsto S$, implied by Lemma 3.8, corresponds to the restriction operator from [Ko1] which is derived by

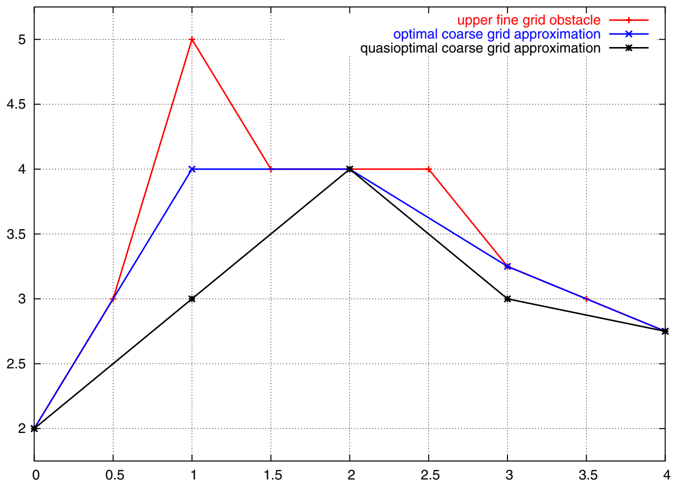


FIG. 3.1. *Continuous piecewise linear upper obstacle function on the fine grid* $[0,4] \cap \mathbb{Z}/2$ *and coarse grid approximations according to Lemma* 3.8 *and Proposition* 3.4, *respectively, on the coarse grid* $[0,4] \cap \mathbb{Z}$.

geometric considerations. It is an improvement of the restriction operator $\hat{r}$ from Remark 3.5 or [Ma] since $\tilde{r}(\tilde{S}) \geq \hat{r}(\tilde{S})$ holds for all $\tilde{S} \in \mathcal{N}_{2,\Delta}$.

In Figure 3.1 a continuous, piecewise linear, upper obstacle function, the optimal coarse grid approximation according to Lemma 3.8 and the coarse grid approximation according to Proposition 3.4 are displayed. The improvement of the simple approximation L_2 is clearly visible. Since the band width of A_k increases with increasing order k, and since the Fourier–Motzkin elimination is only suited for small matrices or for matrices with mainly zero entries [Sj], a different approach must be found to solve the linear optimization problem in the higher order case $k > 2$.

To simplify the notation we define in addition to (3.5) that $a_j := 0$ for $j > k$ and $j < 0$.

THEOREM 3.10 (optimized coarse grid correction (OCGC) scheme). *Let* $\tilde{S} \in \mathcal{N}_{k,\Delta}$ *be given with* $\tilde{S} = \tilde{\mathbf{c}}^T \mathbf{N}_{k,\Delta}$. *Let* $L_k \in \mathcal{N}_{k,T}$ *with* $L_k = \mathbf{q}^T \mathbf{N}_{k,T}$ *be as in* (3.9) *and define*

$$\tilde{b}_j = \tilde{b}_j(c_1, \dots, c_{\lfloor (j+k)/2 \rfloor - 1}) := \tilde{c}_j - \sum_{\nu=1}^{\lfloor (j+k)/2 \rfloor - 1} a_{j+k-2\nu}\, c_\nu \tag{3.16}$$

for $j = 1, \dots, \tilde{n}$ *and* $\tilde{b}_j := \infty$ *for* $j < 1$. *Let* $\hat{b}_{m,i} := \infty$ *for* $m > \tilde{n}$ *or* $m < 1$ *and*

$$\hat{b}_{m,i} = \hat{b}_{m,i}(c_1, \dots, c_{i-1}) := \tilde{c}_m - \sum_{\nu=1}^{i-1} a_{m+k-2\nu}\, c_\nu - \sum_{\nu=i+1}^{\lfloor (m+k)/2 \rfloor} a_{m+k-2\nu}\, q_\nu \tag{3.17}$$

for $i = 1, \dots, n$ *and* $m = 2i - k + 2, \dots, 2i$, *where* $q_j := 0$ *for* $j > n$. *Further, let the* *vector* $\mathbf{c}$ *be recursively defined by*

$$c_i := \min\left\{\frac{\tilde{b}_{2i-k}}{a_0}, \frac{\tilde{b}_{2i-k+1}}{a_1}, \frac{\hat{b}_{2i-k+2,i}}{a_2}, \ldots, \frac{\hat{b}_{2i,i}}{a_k}\right\} \quad \textit{for } i = 1, \ldots, n. \tag{3.18}$$

Then $S = \mathbf{c}^T \mathbf{N}_{k,T} \in \mathcal{N}_{k,T}$ *is a monotone and quasi-optimal coarse grid approximation to* $\tilde{S}$, *i.e.,*

$$L_k(x) \le S(x) \le \tilde{S}(x) \quad \textit{for all } x \in I.$$

Proof. We only consider the case k odd as the other case is analogous.

We first derive conditions which guarantee monotonicity (3.6) of the approximation. Moving all entries $a_{i+k-2j}c_j$ of the inequality system (3.6) except for the rightmost nonzero ones in each row to the right-hand side leads to

$$\begin{pmatrix} a_0 & 0 & & \\ a_1 & 0 & & \\ 0 & a_0 & 0 & \\ 0 & a_1 & 0 & \\ & & \ddots & \\ & & 0 & a_0 \\ 0 & \cdots & 0 & a_1 \end{pmatrix} \begin{pmatrix} c_{\ell+1} \\ c_{\ell+2} \\ \vdots \\ c_n \end{pmatrix} \le \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \\ \tilde{b}_4 \\ \vdots \\ \tilde{b}_{\tilde{n}-1} \\ \tilde{b}_{\tilde{n}} \end{pmatrix} \tag{3.19}$$

with $\ell := \lfloor (k-1)/2 \rfloor$ and the new right-hand side coefficients $\tilde{b}_i$ defined in (3.16). From (3.19) we immediately obtain that the inequality system $A_k\mathbf{c} \le \tilde{\mathbf{c}}$ is satisfied for arbitrary $c_1, \ldots, c_\ell$ if

$$c_i \le \min\left\{\frac{\tilde{b}_{2i-k}}{a_0}, \frac{\tilde{b}_{2i-k+1}}{a_1}\right\} \quad \text{for } i = \ell+1, \ldots, n \tag{3.20}$$

holds.

Second, we derive conditions which ensure quasi-optimality $\mathbf{c} \ge \mathbf{q}$ of the approximation. For an arbitrary $j \in \{\ell+1, \ldots, n\}$ the first inequality of (3.20) and definition (3.16) imply

$$a_0 c_j \le \tilde{b}_{2j-k} = \tilde{c}_{2j-k} - \sum_{\nu=1}^{j-1} a_{2j-2\nu}\, c_\nu.$$

For every $i \in \{1, \ldots, j-1\}$, we therefore obtain the condition

$$a_{2j-2i} c_i \le \tilde{c}_{2j-k} - \sum_{\substack{\nu=1 \\ \nu \ne i}}^{j} a_{2j-2\nu}\, c_\nu.$$

When we determine c_i, we can assume that the c_ν's for $\nu = 1, \ldots, i-1$ are already computed. For the c_ν, $\nu = i+1, \ldots, j$, which are yet to be determined, demanding quasi-optimality $c_\nu \ge q_\nu$ leads to

$$a_{2j-2i} c_i \le \tilde{c}_{2j-k} - \sum_{\nu=1}^{i-1} a_{2j-2\nu}\, c_\nu - \sum_{\nu=i+1}^{j} a_{2j-2\nu}\, q_\nu = \hat{b}_{2j-k,i} \tag{3.21}$$

with $\hat{b}_{j,i}$ defined in (3.17). Analogously we get

$$a_{2j-2i+1}c_i \leq \hat{b}_{2j-k+1,i} \tag{3.22}$$

for $i < j$ using the second inequality of (3.20). Because of $a_m = 0$ for $m > k$, the inequalities (3.21) and (3.22) only apply for $i+1 \leq j \leq i+\ell$ so that we obtain the conditions

$$c_i \leq \min\left\{\frac{\hat{b}_{2i-k+2,i}}{a_2}, \ldots, \frac{\hat{b}_{2i,i}}{a_k}\right\} \tag{3.23}$$

for $i = 1, \ldots, n$. Then both (3.20) and (3.23) are satisfied by defining c_i, $i = 1, \ldots, n$, as in (3.18) which completes the proof. □

Remark 3.11. If one only aims at a coarse grid approximation S which is monotone by construction, one could use the relation (3.20) and replace the inequality by an equality sign. However, in many cases the as-large-as-possible choice of the components c_i according to (3.20) then has to be balanced to preserve monotonicity by very small, maybe even negative components c_j, $j > i$, which leads to undesirable oscillations in the solution. This is avoided by taking in addition the lower bounds into consideration.

Example 3.12. In the case $k = 2$ the recursion (3.18) recovers the direct solution

$$c_i = \min\{2\,\tilde{c}_{2i-2} - c_{i-1},\, \tilde{c}_{2i-1},\, 2\,\tilde{c}_{2i} - q_{i+1}\} \tag{3.24}$$

from Lemma 3.8. For $k = 3$ the recursion (3.18) simplifies to

$$c_i = \min\left\{4\,\tilde{c}_{2i-3} - 3\,c_{i-1}, \tfrac{4}{3}\tilde{c}_{2i-2} - \tfrac{1}{3}c_{i-1}, \tfrac{4}{3}\tilde{c}_{2i-1} - \tfrac{1}{3}q_{i+1}, 4\,\tilde{c}_{2i} - 3\,q_{i+1}\right\}. \tag{3.25}$$

In the case $k = 4$, one obtains

$$\begin{aligned} c_i := \min\big\{&8\tilde{c}_{2i-4} - c_{i-2} - 6c_{i-1}, 2\tilde{c}_{2i-3} - c_{i-1}, \tfrac{4}{3}\tilde{c}_{2i-2} - \tfrac{1}{6}(c_{i-1} + q_{i+1}), \\ &2\tilde{c}_{2i-1} - q_{i+1},\ 8\tilde{c}_{2i} - 6q_{i+1} - q_{i+2}\big\}, \end{aligned} \tag{3.26}$$

where we use the notation that all terms in (3.24)–(3.26) which involve c_j with $j < 1$ or q_j with $j > n$ have to be omitted.

Using (3.2) and exploiting the fact that the number of nonzero terms in each of the sums in the definitions (3.16) and (3.17) is bounded by k, the above algorithm works in optimal complexity.

THEOREM 3.13. *For fixed $k \in \mathbb{N}$, the costs of the OCGC algorithm is restricted by $O(n)$ operations.*

Next, we visualize the effect of our algorithm. In Figure 3.2, one can see a C^1-smooth, piecewise quadratic upper obstacle, the coarse grid approximation obtained by the OCGC algorithm, the coarse grid approximation $L_3 \in \mathcal{N}_{3,T}$ according to Proposition 3.4, and the optimal coarse grid approximation obtained by the simplex algorithm. (Recall, however, that the simplex algorithm does not yield the solution in optimal complexity.) The improvement of the OCGC approximation over the spline L_3 is clearly visible. There is no difference of our OCGC approximation to the optimal coarse grid approximation obtained by the simplex method, except for a slight variation in the interval [0,2]. This difference seems to be caused by boundary effects which has been confirmed in further numerical experiments. As expected, smooth parts of the obstacle are very well approximated, while variations of the obstacle

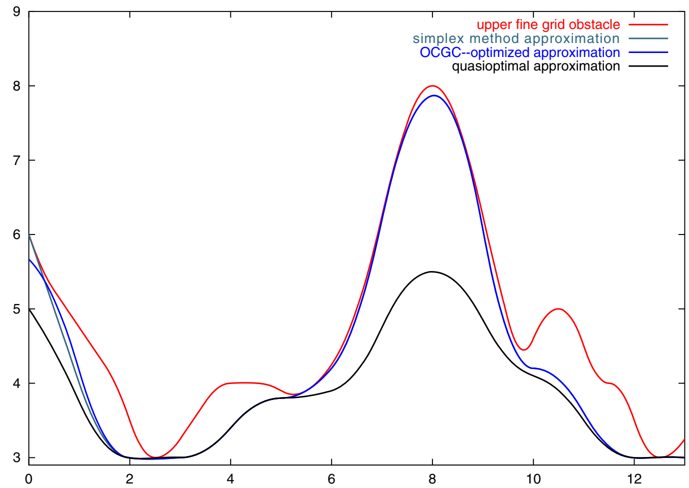


FIG. 3.2. *Right: C^1-smooth, quadratic upper obstacle function on the fine grid $\Delta := [0, 13] \cap \mathbb{Z}/2$ with OCGC-optimized quadratic restriction, the optimal coarse grid approximation obtained by the simplex method and lower quasi–optimal barrier L_3, all three of which are defined on the coarse grid $T := [0, 13] \cap \mathbb{Z}$.*

of higher frequency can only be partly approximated as it is visible in the interval $[10, 12]$. In this example, the control polygon of the B-spline coefficients of the OCGC approximation (which is not displayed here) is partly above the control polygon of the obstacle function, although by construction the OCGC approximation always lies below the obstacle. This indicates that the result of our OCGC algorithm is superior to alternative methods in which monotone approximations are obtained via monotone restrictions of control polygons.

4. Higher spatial dimensions. In the multivariate case $\Omega \subset \mathbb{R}^d$, using (2.12), a d-dimensional spline $S : \Omega \to \mathbb{R}$ of order k can be represented by

$$S(x) = \sum_{i \in \mathbb{I}_c} c_i \, N^{(d)}_{i,k,T}(x) =: \mathbf{c}^T \, \mathbf{N}^{(d)}_{k,T}(x), \quad x \in \Omega, \tag{4.1}$$

with coefficients $\mathbf{c} \in \mathbb{R}^{n^d}$ and indices from $\mathbb{I}_c := \{i \in \mathbb{N}^d : 1 \le i_m \le n, \ m = 1, \ldots, d\}$. The two-scale relation (3.4) attains the multivariate refinement relation

$$N^{(d)}_{i,k,T} = \sum_{j \in J} a^{(d)}_j \, N^{(d)}_{2i-k+j,k,\Delta} \tag{4.2}$$

with the index set $J := \{j \in \mathbb{N}^d : 0 \le j_m \le k \text{ for } m = 1, \ldots, d\}$ and the subdivision coefficients

$$a^{(d)}_j := 2^{(1-k)d} \prod_{\nu=1}^{d} \binom{k}{j_\nu} \quad \text{for } j \in J. \tag{4.3}$$

The extension of Theorem 3.2 then reads as follows.

THEOREM 4.1 (monotone coarse grid approximation). *The spline* $S = \mathbf{c}^T \mathbf{N}_{k,T}^{(d)}$ *is a monotone coarse grid approximation to the upper obstacle* $\tilde{S} = \tilde{\mathbf{c}}^T \mathbf{N}_{k,\Delta}^{(d)}$ *if their B-spline expansion coefficients satisfy the linear inequality system*

$$A_k^{(d)}\,\mathbf{c} \le \tilde{\mathbf{c}} \tag{4.4}$$

with the tensor product matrix $A_k^{(d)} := A_k \otimes \ldots \otimes A_k \in \mathbb{R}^{(\tilde{n}\times n)^d}$ *and* A_k *as in* (3.6).

Proof. The proof follows by the same arguments as in the univariate case, by using the refinement relation (4.2) and applying the multivariate version of Lemma 2.3 to

$$\tilde{S}(x) - S(x) = \sum_{i\in\mathbb{I}_f} (A_k^{(d)}\,\mathbf{c} - \tilde{\mathbf{c}})_i\, N_{i,k,\Delta}^{(d)}(x),$$

where $\mathbb{I}_f := \{i \in \mathbb{N}^d : 1 \le i_m \le \tilde{n},\ m = 1,\ldots,d\}$ using the nonnegativity of (tensor product) B-splines. ∎

Example 4.2. In the special case of C^2-smooth, piecewise cubic ($k = 4$) splines on a two-dimensional domain, the system (4.4) reads

$$\begin{pmatrix} \frac{4}{8}A_4 & \frac{4}{8}A_4 & & & \\ \frac{1}{8}A_4 & \frac{6}{8}A_4 & \frac{1}{8}A_4 & & \\ & \frac{4}{8}A_4 & \frac{4}{8}A_4 & & \\ & \frac{1}{8}A_4 & \frac{6}{8}A_4 & & \\ & & \ddots & \ddots & \\ & & & \frac{6}{8}A_4 & \frac{1}{8}A_4 \\ & & & \frac{4}{8}A_4 & \frac{4}{8}A_4 \end{pmatrix} \begin{pmatrix} c_{1,1} \\ \vdots \\ c_{1,n} \\ \vdots \\ \vdots \\ c_{n,1} \\ \vdots \\ c_{n,n} \end{pmatrix} \le \begin{pmatrix} \tilde{c}_{1,1} \\ \vdots \\ \tilde{c}_{1,\tilde{n}} \\ \tilde{c}_{2,1} \\ \vdots \\ \tilde{c}_{2,\tilde{n}} \\ \vdots \\ \tilde{c}_{\tilde{n},1} \\ \vdots \\ \tilde{c}_{\tilde{n},\tilde{n}} \end{pmatrix}.$$

As all rows in the system (4.4) sum to one we immediately obtain from Theorem 4.1 the following generalization of Proposition 3.4.

PROPOSITION 4.3. *The spline* $L_k := \mathbf{q}^T\, \mathbf{N}_{k,T}^{(d)}$ *with expansion coefficients*

$$q_i := \min\{\tilde{c}_j :\ 2i_m - k \le j_m \le 2i_m,\ m = 1,\ldots,d\} \quad \text{for } i \in \mathbb{I}_c \tag{4.5}$$

(leaving out $\tilde{c}_j$ *in the right-hand side if* $j_m < 1$ *or* $j_m > \tilde{n}$*) is a monotone coarse grid approximation to the obstacle function* $\tilde{S} = \tilde{\mathbf{c}}^T \mathbf{N}_{k,\Delta}^{(d)}$.

In the special case $k = 2$ Fourier–Motzkin elimination can be applied to the inequality system (4.4) with the constraint $\mathbf{c} \ge \mathbf{q}$ as in the univariate case to obtain Lemma 3.8 for arbitrary d.

LEMMA 4.4 (direct solution for hat functions). *Define the sum* $s_{\ell,i}$ *of all neighboring coarse grid coefficients* c_j *to a given fine grid coefficient* $\tilde{c}_\ell$ *except* c_i *by*

$$s_{\ell,i} := \sum_{\{j\in\mathbb{I}_c:\, j\ne i,\, |\theta_{j+1}-\tilde{\theta}_{\ell+1}|<H\}} c_j \quad \text{for } \ell \in \mathbb{I}_f,\ i \in \mathbb{I}_c,$$

with the Euclidean distance $|\cdot|$ *and the mesh size* H *of the coarse grid. Define* $\bar{s}_{\ell,i}$ *as* $s_{\ell,i}$ *with the modification that all coefficients* c_j *in the sum which are not yet known* *are replaced by* q_j *given from* (4.5). *Then, for* $k = 2$, *a monotone and quasi-optimal coarse grid approximation is recursively given by*

$$c_i := \min\{2^{\sum_{m=1}^{d} |\ell_m-(2i_m-1)|}\, \tilde{c}_\ell - \bar{s}_{\ell,i} : \; 2i_m - 2 \le \ell_m \le 2i_m \quad \textit{for} \quad m = 1,\dots,d\}$$

for $i \in \mathbb{I}_c$, *leaving out* $\tilde{c}_j$ *in the right-hand side if* $j_m < 1$ *or* $j_m > \tilde{n}$.

To improve the approximation from Proposition 4.3 in the case $k > 2$, the OCGC algorithm can be applied recursively with respect to the dimension d as follows.

THEOREM 4.5 (optimized coarse grid correction (OCGC) scheme for $d > 1$). *The OCGC algorithm applied dimension-recursively to the multivariate inequality system* (4.4) *provides in optimal complexity of* $O(n^d)$ *arithmetic operations a coarse grid approximation* S *which satisfies the monotonicity and quasi-optimality condition*

$$L_k(x) \le S(x) \le \tilde{S}(x) \quad \textit{for all } x \in \Omega.$$

Proof. We provide the proof for the case $k = 3$; the other cases follow immediately by exchanging the system matrix $A_3^{(d)}$ by $A_k^{(d)}$. The $(n \times \tilde{n})^d$ tensor product matrix $A_3^{(d)}$ can be written as a $n \times \tilde{n}$ matrix of $(n \times \tilde{n})^{(d-1)}$ tensor product matrices

$$A_3^{(d)} := \begin{pmatrix} \frac{3}{4}A_3^{(d-1)} & \frac{1}{4}A_3^{(d-1)} & & & & & \\ \frac{1}{4}A_3^{(d-1)} & \frac{3}{4}A_3^{(d-1)} & & & & & \\ & \frac{3}{4}A_3^{(d-1)} & \frac{1}{4}A_3^{(d-1)} & & & & \\ & \frac{1}{4}A_3^{(d-1)} & \frac{3}{4}A_3^{(d-1)} & & & & \\ & & & \ddots & \ddots & & \\ & & & & & \frac{3}{4}A_3^{(d-1)} & \frac{1}{4}A_3^{(d-1)} \\ & & & & & \frac{1}{4}A_3^{(d-1)} & \frac{3}{4}A_3^{(d-1)} \end{pmatrix}.$$

Accordingly, we define $\mathbf{c}_{i\bullet} := (c_{ij})_{i=1,\dots,n,\, j\in\mathbb{N}^{(d-1)}:\, 1\le j_l\le n} \in \mathbb{R}^{n^{(d-1)}}$. Applying the OCGC algorithm for $d = 1$ (3.25) to this formulation, we obtain for $i = 1,\dots n$ the conditions

$$\begin{aligned} A_3^{(d-1)}\mathbf{c}_{i\bullet} \le \min\{ & 4\,\tilde{c}_{2i-3\bullet} - 3A_3^{(d-1)}\,c_{i-1\bullet}, \tfrac{4}{3}\tilde{c}_{2i-2\bullet} - \tfrac{1}{3}A_3^{(d-1)}c_{i-1\bullet}, \\ & \tfrac{4}{3}\tilde{c}_{2i-1\bullet} - \tfrac{1}{3}A_3^{(d-1)}q_{i+1\bullet}, 4\,\tilde{c}_{2i\bullet} - 3A_3^{(d-1)}\,q_{i+1\bullet}\} \end{aligned}$$

for $\mathbf{c}_{i\bullet} \in \mathbb{R}^{n^{(d-1)}}$. Each of these conditions can again be reduced by one dimension by a further application of the OCGC algorithm until the whole system is solved. As the complexity of the OCGC algorithm is $O(n)$, the overall complexity is given by $O(n^d)$ arithmetic operations. ∎

Example 4.6. In the case $k = 3$ and $d = 2$ the multivariate OCGC algorithm is defined as follows:

(1) For given $\tilde{\mathbf{c}} \in \mathbb{R}^{\tilde{n}^2}$ determine $\mathbf{q} \in \mathbb{R}^{n^2}$ by (4.5).

(2) For $i = 1,\dots,n$

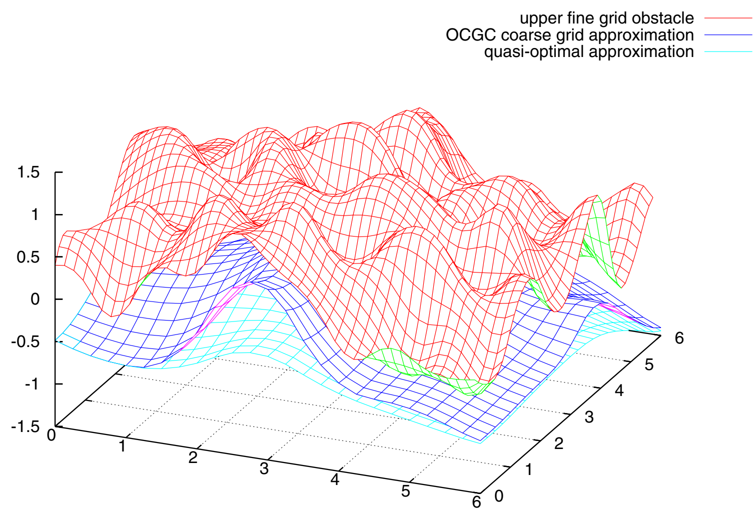


FIG. 4.1. *Two-dimensional C^1-smooth, quadratic, upper obstacle function defined on fine grid* $[0,6]^2 \cap (\mathbb{Z}/2)^2$ *and coarse grid approximations from Proposition* 4.3 *(quasi-optimal) and Theorem* 4.5 *(OCGC) on the coarse grid* $[0,6]^2 \cap \mathbb{Z}^2$.

define $\mathbf{g} \in \mathbb{R}^n$ by $g_j := q_{i,j}$ and $\mathbf{f} \in \mathbb{R}^{\tilde{n}}$ by $f_j := \min\{4\,\tilde{c}_{2i-3,j} - 3(A_3\,c_{i-1}\bullet)_j,$

$$\tfrac{4}{3}\tilde{c}_{2i-2,j} - \tfrac{1}{3}(A_3 c_{i-1}\bullet)_j, \tfrac{4}{3}\tilde{c}_{2i-1,j} - \tfrac{1}{3}(A_3 q_{i+1}\bullet)_j, 4\,\tilde{c}_{2i,j} - 3(A_3\,q_{i+1}\bullet)_j\},$$

solve the univariate problem $A_3\mathbf{e} \le \mathbf{f}, \quad \mathbf{e} \ge \mathbf{g}$ by the 1d-OCGC Algorithm,

set $c_{i,j} := e_j$.

The splines which correspond to the coefficient vector $\mathbf{q}$ and $\mathbf{c}$ from Proposition 4.3 and Theorem 4.5, respectively, are displayed in Figure 4.1 for a given upper obstacle function defined on the fine grid $[0,6]^2 \cap (\mathbb{Z}/2)^2$.

The resulting MMG method in the multivariate case can now be implemented by adding the projection operator (2.11) and the obstacle approximation from Theorem 4.5 to a standard multigrid method. The standard multigrid method for tensor products of higher order B-splines is described, e.g., in [Hö, HRW] for the case $d > 1$.

5. Convergence theory for B-spline-based MMG methods. It is shown in [Ko1] that MMG methods are globally convergent and asymptotically reducing to a linear subspace correction method, provided nodal basis functions and monotone and quasi-optimal restriction operators $\tilde{r}$ are used. Because of the lack of such restriction operators for smooth functions, the MMG method has so far been restricted to hat functions. Using B-splines as basis functions, we have already transferred the scheme to functions of general smoothness in section 2. Suitable restriction operators have been constructed in sections 3 and 4. We have established in the extended version of this paper [HzK] that all convergence results from [Ko1] can be transferred to B-spline basis functions, using their expansion coefficients instead of function values.

6. Numerical example. To present a numerical example from the area of Mathematical Finance, we choose the domain $\Omega_{\mathcal{L}} := \mathbb{R}^+ \times [0,T)$, the differential

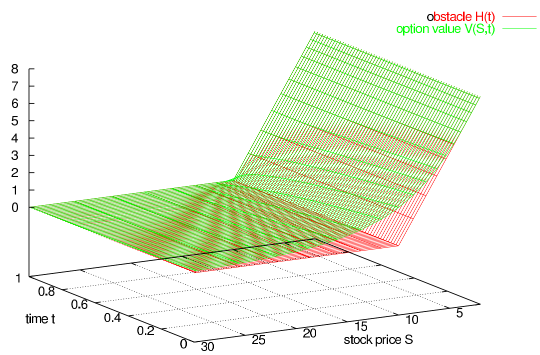


FIG. 6.1. *Solution $V(S, t)$ of the linear complementary problem* (6.2).

operator

$$\mathcal{L} := \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + rS\frac{\partial}{\partial S} - r, \tag{6.1}$$

and the function $\mathcal{H}(S) := (K - S)^+$. We consider the linear complementary problem to find $V = V(S, t) \in H^1(\Omega_{\mathcal{L}})$, such that

$$\begin{aligned} [(\mathcal{L}\,V)(S,t)]\,(V(S,t) - \mathcal{H}(S)) &= 0, \\ (\mathcal{L}\,V)(S,t) &\leq 0, \\ V(S,t) &\geq \mathcal{H}(S) \end{aligned} \tag{6.2}$$

holds for all $(S, t) \in \Omega_{\mathcal{L}}$, with boundary data $V(S, t) = 0$ for $S \to \infty$, $V(S, t) = \mathcal{H}(S)$ for $S \to 0$ and final data $V(S, T) = \mathcal{H}(S)$ for $S \in \mathbb{R}^+$.

As it is shown in [WHD], the solution V describes the fair value of an *American put option* with strike price K and maturity T which depends on an underlying stock with value S and volatility σ. No analytical solution is known for the problem (6.2) so that one has to resort to numerical solution schemes. In the numerical experiments we used for the linear complementary problem (6.2), the parameters $K = 10$ for the strike price, $T = 1$ for maturity, $\sigma = 0.6$ for volatility, and $r = 2.5\%$ for the interest rate. The numerical solution V and the obstacle function $\mathcal{H}$ are displayed in Figure 6.1 in the case of $M = N = 64$ grid points in space and time.

If the obstacle function is set to minus infinity, the solution V describes the fair value of a *European put option* (see [WHD]). In that case an analytical solution is known and given by the famous Black–Scholes formula; see [BS].

Using a Crank–Nicolson finite difference scheme for the time discretization and at least continuous piecewise finite elements for the space discretization, the method converges quadratically. Employing higher order finite element functions, the derivatives of the solution V which provide important hedge parameters in the option pricing context can be determined by direct differentiation of the basis functions. Using B-spline

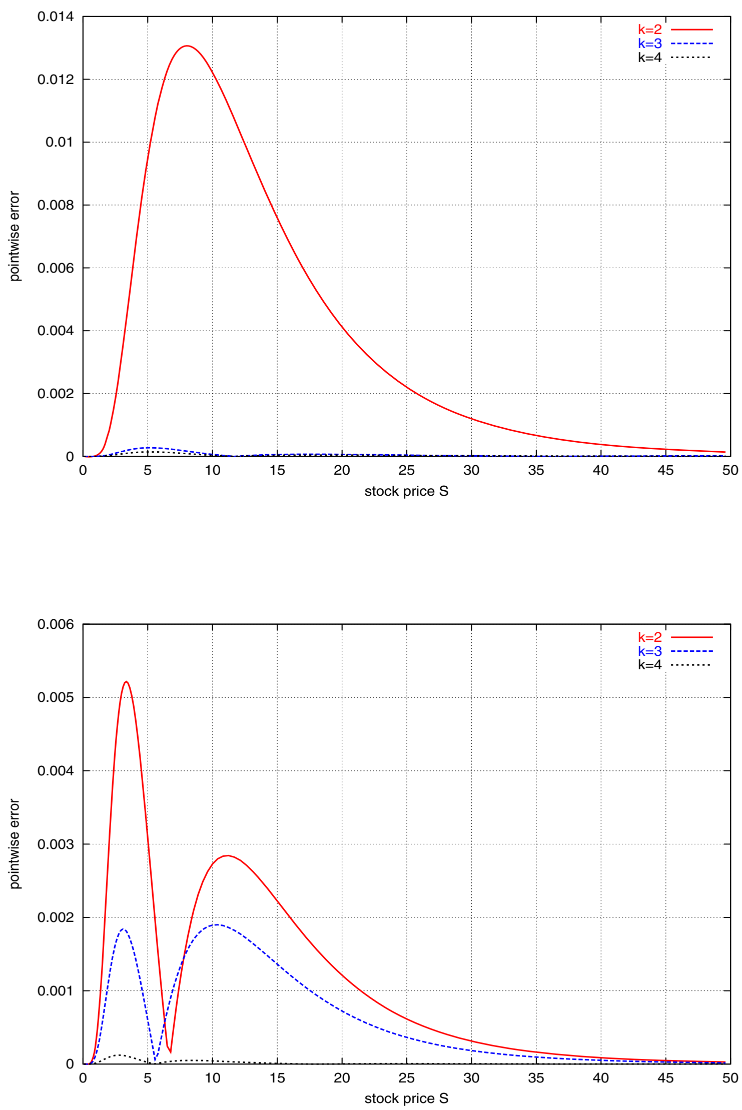


FIG. 6.2. *Comparison of pointwise error for* Delta *and* Gamma *at time* $t = 0$ *for orders* $k = 2, 3, 4$ *and* $N = M = 275$.

bases of order k we obtain all derivatives up to the $(k-2)$th derivative in quadratic convergence. In particular, pointwise derivatives, the so-called Greek letters, can be computed up to high accuracy. These results, as well as extensive discussion, can be found in [Hz]. As an illustration of the impressible difference a variable order k may offer, we display in Figure 6.2 only the pointwise errors of *Delta*:= $\frac{\partial V}{\partial S}$ and *Gamma*:= $\frac{\partial^2 V}{\partial S^2}$.

In view of this application, we would like to point out that our higher order MMG method could also be applied to the valuation of basket options, at least for small

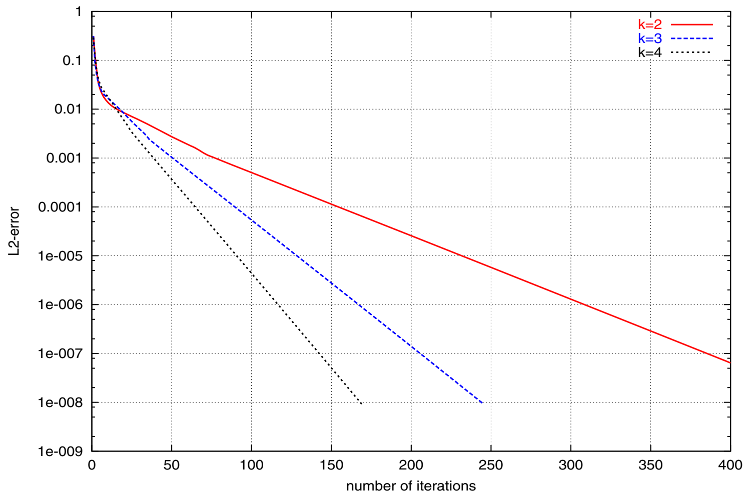


FIG. 6.3. *PSOR–iteration history of one time step with* $M = 256$.

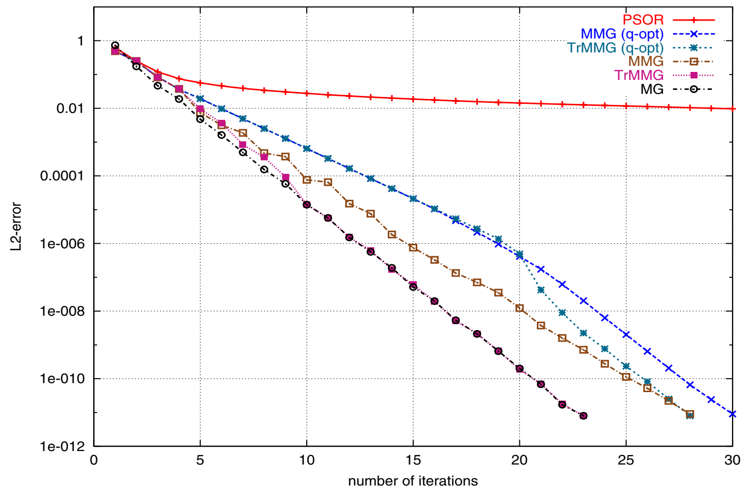
baskets with $d = 2$ or $d = 3$. Similar to the univariate case, the multivariate Black–Scholes equation can be transformed into a multivariate heat diffusion problem, as shown in [RW].

6.1. Convergence behavior of Gauss–Seidel and MMG schemes. In the following, only one time step of problem (6.2) is considered to analyze the performance of the multigrid scheme. In Figure 6.3, the iteration errors of the projected Gauss–Seidel scheme are displayed for different orders k. The impact of the order k is clearly visible. Next we compare the convergence behavior of the following methods:

PSOR	Projected Gauss–Seidel scheme,
MMG	Monotone multigrid method with optimized approximation of the obstacle according to Lemma 3.8 and Theorem 3.10,
TrMMG	Truncated version of the monotone multigrid method [Ko1] with optimized approximation of the obstacle according to Lemma 3.8 and Theorem 3.10,
MMG (q-opt)	Monotone multigrid method with simple approximation of the obstacle according to Proposition 3.4,
TrMMG (q-opt)	Truncated version of the monotone multigrid method with simple approximation of the obstacle according to Proposition 3.4,
MG	Linear multigrid method applied to the unrestricted problem.

To analyze the influence of the order k on the convergence behavior, the case $k = 2$ is systematically compared to the case $k = 3$. For $k > 3$ similar results are expected. In the experiments of the finance parameters used in the previous section, the finest level $L = 7$ and a random initial guess have been chosen. To make sure that the iteration does not terminate too early, we have selected independently of the discretization error the stopping criterion

$$\|u_L^{\nu+1} - u_L^{\nu}\|_\infty \leq 10^{-12},$$



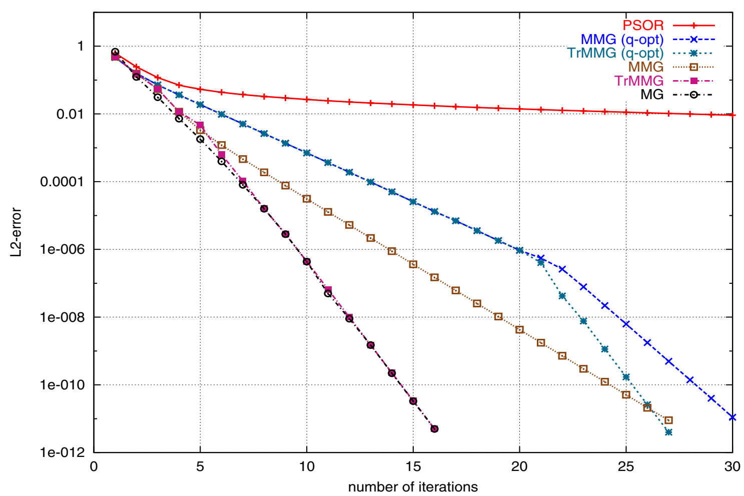


FIG. 6.4. *Iteration history for hat functions ($k = 2$) (top) and for C^1-smooth basis functions ($k = 3$) (bottom).*

where u_L^ν denotes the νth iterate on the finest grid L.

The numerical results are summarized in Figure 6.4 and Table 6.1. In the third column in Table 6.1, the number ν_0 of iterations needed to identify the contact set $K^\bullet(u_L)$ is displayed. In the next column $\sharp$It., we list the number of iterations which are needed to solve the problem up to machine accuracy. To compare the costs of the schemes, we employ the definition of a work unit (WU) from [BC]. A work unit $\mathtt{WU} = \mathtt{WU}_L$ denotes the costs of one iteration step of the projected Gauss–Seidel scheme on the finest grid L. The costs $\mathtt{WU}_\ell$ of one iteration step on level $\ell \leq L$ is then given

	Scheme	1 smoothing step			2 smoothing steps		
		ν_0	♯ It.	♯WU	ν_0	♯ It.	♯ WU
$k=2$	PSOR	134	403	403	—	—	—
	MMG (q-opt)	6	30	59.06	5	21	82.69
	TrMMG (q-opt)	7	28	55.13	5	17	66.94
	MMG	7	28	55.13	5	14	55.13
	TrMMG	7	23	45.28	5	13	51.19
$k=3$	PSOR	103	447	447	—	—	—
	MMG (q-opt)	5	31	61.03	4	20	78.75
	TrMMG (q-opt)	6	27	53.16	4	17	66.94
	MMG	5	27	53.16	4	14	55.13
	TrMMG	5	16	31.5	4	11	43.31

TABLE 6.1
Number of iterations needed to identify the contact set and to compute the solution up to machine accuracy and the cost in work units.

by

$$\mathtt{WU}_\ell = 2^{L-\ell}\,\mathtt{WU}_L.$$

The number of work units which is needed to reach the stop criteria is displayed in the last column ♯WU in Table 6.1.

The numerical results show that already one or two smoothing steps are sufficient with regard to cost and accuracy. In comparison to the Gauss–Seidel relaxation, the cost is substantially reduced in the multigrid schemes. The truncated versions TrMMG and TrMMG (q-opt) converge in all cases faster than the standard versions MMG or MMG (q-opt). Moreover, multigrid methods with an optimized approximation of the obstacle according to Lemma 3.8 or Theorem 3.10 converge faster than the simple approximations according to Proposition 3.4. For hat functions, this corresponds to the results in [Ko1]. For the higher order case, this indicates the quality of the OCGC approximations from section 3.4. The contact set is identified correctly by all methods within only a few iterations.

Considering the above results within the time discretization when solving the instationary problem, we wish to point out that the average number of iterations per time step is much smaller. This is due to the fact that the solution of the previous time step serves as a good initial guess. Therefore, we can expect that the asymptotic phase dominates the convergence behavior of the multigrid scheme. The asymptotic multigrid rates are discussed in the following section.

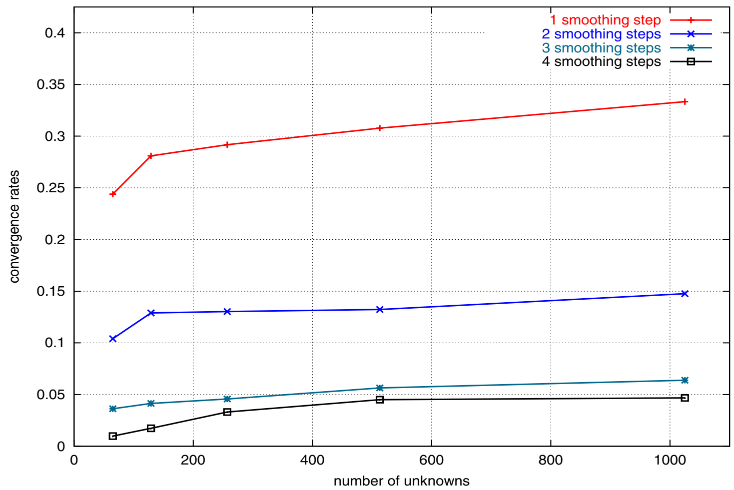
6.2. Multigrid convergence rates. The convergence rate ρ_ℓ of a multigrid scheme with $\ell+1$ levels is given by

$$\|u_\ell^{\nu+1} - u_\ell\|_{\ell_2} \le \rho_\ell \|u_\ell^\nu - u_\ell\|_{\ell_2}.$$

Here $u_\ell \in S_\ell$ denotes the exact solution and $u_\ell^\nu \in S_\ell$ the approximate solution in the νth iteration step. A scheme is said to have multigrid convergence if ρ_ℓ is bounded independently of the grid size by a constant $\rho_\infty < 1$.

The asymptotic convergence rates are estimated for the V-cycle of the truncated version TrMMG with $\ell+1$ levels according to

$$\rho_\ell \approx \frac{\|u_\ell^{\nu^*+1} - u_\ell^{\nu^*}\|_{\ell_2}}{\|u_\ell^{\nu^*} - u_\ell^{\nu^*-1}\|_{\ell_2}}.$$



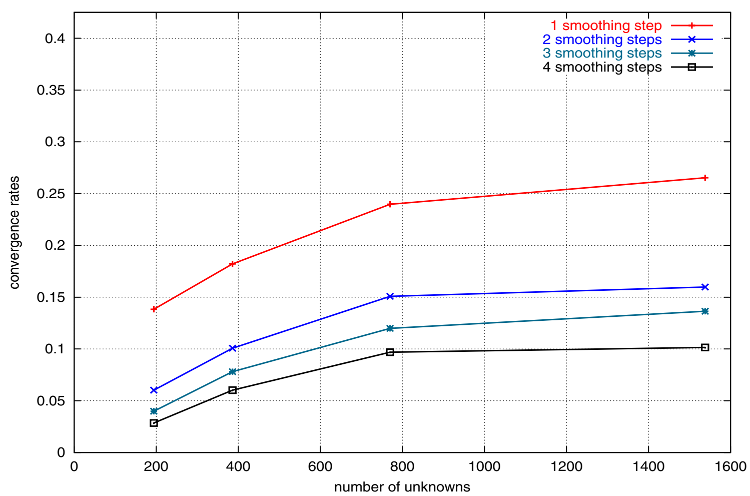


FIG. 6.5. *Asymptotic convergence rates for the case $k = 2$ (left) and $k = 3$ (right) depending on the number M of unknowns.*

Here ν^* is chosen such that $\|u_\ell^{\nu^*+1} - u_\ell^{\nu^*}\|_{\ell_2} \leq 10^{-12}$. In Figure 6.5 the results are displayed on the left-hand side for continuous, piecewise linear basis functions and on the right-hand side for C^1-smooth, piecewise quadratic basis functions. We recover the favorable convergence rates of standard multigrid schemes which are bounded in our case by $\rho_\infty \approx 0.31$ ($k = 2$) and $\rho_\infty \approx 0.27$ ($k = 3$) in the case of only one smoothing step on each refinement level.

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