Title: Multiresolution Methods

Name: Angela Kunoth

Affil./Addr.: Institut für Mathematik, Universität Paderborn, Germany

E-mail: kunoth@math.uni-paderborn.de

http://www2.math.uni-paderborn.de/ags/kunoth

Multiresolution Methods

Short Definition

Multiresolution (or multiscale) methods decompose an object additively into terms on different scales or resolution levels. The object can be given explicitly, e.g., as time series or image data, or implicitly, e.g., as the solution of a partial differential equation.

Description

Many physical problems exhibit characteristic features at multiple temporal and/or spatial scales. The goal of multiresolution methods is to decompose the object of interest into objects resolving these scales, for the purpose of analysis, approximation, compression, processing etc. Typical examples are measurement signals or time series, described as univariate given functions f living on a finite interval $[0,T] \subset \mathbb{R}$. The goal is to find a decomposition

$$f(t) = \sum_{j=0}^{\infty} g_j(t), \qquad t \in [0,T], \tag{1}$$

where the index $j \in \mathbb{N}$ stands for the *scale* or *resolution* and indicates for growing j finer scales. For a time series, f is represented by point values on a discrete grid (which may be viewed as a *single-scale representation* of the data), and the series in (1) is finite. A synthetic function consisting of three components g_0, g_1, g_2 is shown in Fig. 1.

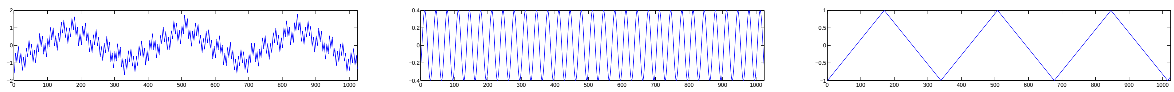

Fig. 1. Synthetic function f (left), additively composed from a sine wave g_1 (middle) and two piecewise linear continuous functions of different resolutions, one of them g_0 (right).

Classical decompositions (1) assume that the multiscale components g_j are of a particular form and all of the same shape: in Fourier analysis, these are the Fourier components $g_j(t) = a_j \exp(i\omega_j t)$ with prescribed frequencies ω_j and constant amplitudes a_j to be computed from f by, for example, the *Fast Fourier Transform*. In the example in Fig. 1, the component g_1 is of this form. Other examples are hierarchical decompositions where the g_j's are assumed to be of the form

$$g_j(t) = \sum_{k \in K} d_{j,k}\, \psi_{j,k}(t). \tag{2}$$

Here $\psi_{j,k}$ are prescribed functions, typically generated from a single translated and dilated function of local support; the additional index k represents the *location*. Standard cases for $\psi_{j,k}$ are piecewise polynomials, B-splines, or finite elements. These would be appropriate to represent the components g_0 and g_2 in Fig. 1. In these cases, one can compute the expansion coefficients $d_{j,k}$ by interpolation or projection from the given data f, and (1) together with (2) results in a *hierarchical* or *multiscale data representation*. If the collection of all functions $\psi_{j,k}$ for all levels j and all locations k satisfy additional conditions (like constituting a Riez basis for the underlying function space, often the Lebesgue space $L_2(0,T)$), one calls this a *wavelet decomposition*. The construction of wavelets themselves is typically based on the concept of *multiresolution analysis* of a separable Hilbert space [Mallat(1999)]. For given uniformly distributed data f, the expansion coefficients $d_{j,k}$ can be determined by the *Fast Wavelet Transform* [Daubechies(1992), Mallat(1999)]. Thus, the computation of these types of multiresolution decompositions rely on applying *linear* transformations. In case of non-uniformly spaced data, the application of these transforms often resorts to the uniform grid case.

For data in more than one dimension like images, one typically applies these transforms for each coordinate direction. The resulting multiscale or hierarchical decompositions are then used for image analysis and compression or the fast processing of surfaces.

For given data exhibiting nonlinear and nonstationary features on possibly nonuniform grids, a more recent method is based on a data-adaptive iterative process, leading to the so-called *empirical mode decomposition* [Huang and Shen(2005)].

If the object in question is to be determined as the solution u of an operator equation $F(u) = g$, e.g., a partial differential or integral equation on infinite Banach spaces, the principle of finding a decomposition (1) is the same, enhanced to a large extent by the difficulty to solve the equation. The type of equation dominates the discretization and solution approach. One uses the terminology 'multiresolution method' to describe the following methodologies:

(i) homogenization and multiscale modeling to resolve multiple scales the solution exhibits;

(ii) multigrid methods (preconditioning, i.e., using multiple scales for computational speedup, developing fast solvers for linear systems of equations stemming from discretization of, e.g., elliptic partial differential equatios (PDEs));

(iii) compression of integral operators and computation of high-dimensional integrals (appearing, e.g. in quantum chemistry);

(iv) a-posteriori adaptive methods to compute the solution u, starting from a coarse approximation to progressively include finer scales resolving singularities in data and/or domain during the computations.

An extensive source of discussion of the points (ii)–(iv) are [Cohen(2003), Dahmen(1997)] and the invited surveys collected in [DeVore and Kunoth(2009)]: wavelet preconditioning in the context of (ii) in [Kunoth(2009)]; (iii) based on wavelets in [Harbrecht and Schneider(2009)] and by exponential sums in [Braess and Hackbusch(2009)];

(iv) for hyperbolic conservation laws discretized by finite volume schemes in [Müller (2009)]; for elliptic PDEs discretized by finite elements in [Nochetto et al(2009)] (see also adaptive mesh refinement) and by wavelets in [Stevenson(2009)], and the development of multilevel schemes for systems of PDEs in [Xu et al(2009)].

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